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Journal of Functional Analysis 262 (2012) 4341–4376

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Singular invariant trilinear forms and covariant (bi-)differential operators under the conformal group

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Received 7 September 2011; accepted 24 February 2012

Available online 13 March 2012

Communicated by P. Delorme

Abstract

The residues of the meromorphic family of conformally invariant trilinear forms on the sphere (constructed in Clerc and Ørsted, in press, [2]) are computed. Their expression involves conformally covariant differential and bidifferential operators. For the latter, new formulæ are obtained.

Keywords: Invariant trilinear forms; Conformally covariant differential operators; Bernstein–Sato identity

0. Introduction

Let S be the unit sphere in a Euclidean space of dimension n $(n \geqslant 3)$.[1] Let $G = SO_0(1,n)$ be the (connected component of) the Lorentz group, acting on S by conformal transformations. Let $(\pi_\lambda)_{\lambda\in\mathbb{C}}$ be the (non-unitary) spherical principal series of G, realized on $\mathcal{C}^\infty(S)$.

Let $\lambda_1, \lambda_2, \lambda_3$ be three complex numbers. A continuous trilinear form $\mathcal{K}$ on $\mathcal{C}^\infty(S)\times\mathcal{C}^\infty(S)\times\mathcal{C}^\infty(S)$ is said to be *conformally invariant* with respect to $\pi_{\lambda_1}\otimes\pi_{\lambda_2}\otimes\pi_{\lambda_3}$, if, for any three functions $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$,

$$\mathcal{K}\big(\pi_{\lambda_1}(g)f_1, \pi_{\lambda_2}(g)f_2, \pi_{\lambda_3}(g)f_3\big) = \mathcal{K}(f_1, f_2, f_3)$$

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[1] The case $n = 2$ could be treated along the same lines, but there are some differences, which would require separate statements. See [15] for a study of this case.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2012.02.021

for any g in G. These trilinear forms have been investigated in a previous work of the second author in collaboration with B. Ørsted (see [2]). Generically, for $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ in $\mathbb{C}^3$, there is a unique (up to a multiple) such invariant trilinear form. Viewing the trilinear form as a distribution on $S \times S \times S$, it has a smooth density on the open set

$$\big\{(x_1, x_2, x_3) \in S \times S \times S;\ x_1 \neq x_2,\ x_2 \neq x_3,\ x_3 \neq x_1\big\},$$

given by

$$k_{\boldsymbol{\alpha}}(x_1, x_2, x_3) = |x_1 - x_2|^{\alpha_3} |x_2 - x_3|^{\alpha_1} |x_3 - x_1|^{\alpha_2}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a triplet of complex numbers, uniquely determined by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ (see (7)). The corresponding distribution $\mathcal{K}_{\boldsymbol{\alpha}}$ is defined by meromorphic continuation, and has simple poles along certain planes in $\mathbb{C}^3$. There are two different types of poles, called respectively of type I and of type II (see (9) and (10) for a definition). The present paper deals with the *residues* of this meromorphic family. The residues are distributions supported on proper submanifolds of $S \times S \times S$.

The computation of the residues of type I follows the classical approach to the meromorphic continuation of f^s as used by Gelfand and Shilov [8], which uses appropriate changes of variables (typically polar coordinates)[2] leading to a meromorphic continuation problem for an elementary function of one variable. In our case, the expression of the residues involves *conformally covariant differential operators* on S.

The computation of the residues of type II is more difficult. First, we have to replace the compact realization of the representations π_λ by their *noncompact realization*, which transfers our problem to the flat setting (S being replaced by a Euclidean space of the same dimension). Next, we use a *Bernstein–Sato identity*, which is the tool used in the second proof of the general meromorphic continuation theorem for f^s. After computing the residue for the “first” plane of poles by elementary techniques, the Bernstein–Sato identity allows, by induction, to compute the residues along the other planes of poles. In our case, the expression of the residues involves *conformally covariant bidifferential operators* on S.

The computations of the residues at poles of type II lead to a new formula for conformally covariant bidifferential operators (generically, there is, up to a constant, only one such operator, see [17]), which might be of independent interest.

Finally, the proof of the Bernstein–Sato identity is based on properties of the classical Knapp–Stein intertwining operators and the generic uniqueness theorem for trilinear invariant forms. It is tempting to look for generalizations of such identities in the context of other semi-simple Lie groups and/or other representations.

The paper is organized as follows. Section 1 presents the geometric situation, the representations π_λ and recalls the main result of [2] on the conformally invariant trilinear forms. Section 2 contains the computation of the residues at pole of type I. First, we construct new (singular) trilinear invariant forms associated to covariant differential operators, using a meromorphic continuation procedure. Then we show that the residues at poles of type I coincide (up to a constant) with such a singular trilinear form. Section 3 is devoted to poles of type II. After presenting the noncompact picture, we give an abstract argument showing that the residues are essentially given

[2] A pedestrian variant of Hironaka’s desingularization theorem used by Atiyah to give a proof of the meromorphic continuation theorem for f^s.

by a covariant bidifferential operator. Next, we compute the residue at a pole in the "first" plane of poles. The next Subsection 3.3 contains the proof of a Bernstein–Sato identity for the kernel of the generic trilinear form. The identity is then used to find an expression for the residue at a pole in the other planes of poles. The associated covariant bidifferential operator is explicitly given by an inductive formula. Section 4 contains final remarks and suggests further investigations. The paper ends with Appendix A on conformally covariant differential operators. We thought the presentation of this classical material could be useful to the reader. First, the statements concerning these operators are needed in Section 2. Second, we introduce these operators as residues (of the Knapp–Stein intertwining operators), following exactly the same procedure used in Section 3: computation of the residue at the "first" pole by elementary technique, proof of a Bernstein–Sato identity and computation of the residue at other poles by repeated uses of the Bernstein–Sato identity, so that Appendix A can also be read as a preparation for Section 3.

1. Conformally invariant trilinear forms on the sphere (generic case)

1.1. The geometric context

Let $S = S^{n-1}$ be the unit sphere in $\mathbb{R}^n$,

$$S = \left\{x = (x_1, x_2, \dots, x_n);\ |x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{\frac{1}{2}} = 1\right\}$$

equipped with the Riemannian metric naturally induced from the Euclidean structure of $\mathbb{R}^n$. We assume $n \geqslant 3$, although most of the statements are (or could be made) valid for $n = 2$ (see [2] for similar remarks).

The group $K = SO(n, \mathbb{R})$ operates transitively on S by isometries. Introduce the base point $\mathbf{1} = (1, 0, \dots, 0)$ in S. The stabilizer of $\mathbf{1}$ in K is the subgroup $M \simeq SO(n-1)$ and $S \simeq K/M$ is in particular a compact Riemannian symmetric space.

However, our interest is in the conformal geometry of the sphere, for which another model of the sphere is more fitted. Let $\mathbb{R}^{1,n}$ be the real vector space of dimension $n+1$ equipped with the Lorentzian quadratic form

$$[y, y] = y_0^2 - \left(y_1^2 + \cdots + y_n^2\right)$$

and let $\mathcal{S}$ be the set of all isotropic lines in $\mathbb{R}^{1,n}$, viewed as a closed submanifold of the real n-dimensional projective space. Then the mapping

$$S \to \mathcal{S}, \qquad x \mapsto \mathbb{R}(1, x)$$

is a 1–1 correspondence, which is easily seen to be a diffeomorphism. The group $G = SO_o(1, n)$ operates naturally on $\mathcal{S}$, and this action can be transferred to an action of G on S. The group K can be viewed as a closed subgroup of G and is a maximal compact subgroup of G. The stabilizer of $\mathbf{1}$ in G is a parabolic subgroup P of G, such that $S \simeq G/P$.

The action of G turns out to be *conformal*, that is, for any g in G and x in S, the differential $Dg(x)$ satisfies

$$\left|Dg(x)\xi\right| = \kappa(g, x)|\xi|, \tag{1}$$

for any ξ in the tangent space T_xS, where $\kappa(g,x)$ is a positive constant, called the *conformal factor* of g at x. The conformal factor $\kappa(g,x)$ is a smooth function of both g and x, which satisfies the following *cocycle property*:

$$\forall g_1, g_2 \in G,\ \forall x \in S, \quad \kappa(g_1g_2, x) = \kappa\big(g_1, g_2(x)\big)\kappa(g_2, x). \tag{2}$$

Let g be an element of G. As the dimension of the tangent space T_xS is $n-1$, the Jacobian of g at x (with respect to the Euclidean measure on S) is given by

$$j(g,x) = \kappa(g,x)^{n-1}.$$

The corresponding change of variable formula is

$$\int_S f\big(g^{-1}(x)\big)\,d\sigma(x) = \int_S f(y)\kappa(g,y)^{n-1}\,d\sigma(y), \tag{3}$$

where $d\sigma$ is the Lebesgue measure on S.

The Euclidean distance, restricted to $S \times S$, satisfies an important covariance property under the action of G.

Proposition 1.1. *Let g be in G and x, y in S. Then*

$$\big|g(x) - g(y)\big| = \kappa(g,x)^{\frac{1}{2}}|x-y|\kappa(g,y)^{\frac{1}{2}}. \tag{4}$$

Notice that (1) can be viewed as the infinitesimal form of (4).

For α a complex parameter, the formula

$$k_\alpha(x,y) = |x-y|^\alpha$$

defines a smooth function outside of the diagonal of $S \times S$. For f a function in $\mathcal{C}^\infty(S)$ we define (whenever it makes sense) $K_\alpha f$ by the formula

$$K_\alpha f(x) = \int_S k_\alpha(x,y) f(y)\,d\sigma(y).$$

1.2. The spherical principal series

To the conformal action of G on S is associated a family of representations (usually called the spherical principal series). For convenience set $\rho = \frac{n-1}{2}$. Now let λ be any complex number and define for any g in G and f in $\mathcal{C}^\infty(S)$

$$\pi_\lambda(g)f(x) = \kappa\big(g^{-1},x\big)^{\rho+\lambda} f\big(g^{-1}(x)\big)$$

for x in S. Thanks to the cocycle property of κ (see (2)), π_λ is a representation of G on the space $\mathcal{C}^\infty(S)$, which is continuous for the usual topology on $\mathcal{C}^\infty(S)$. For a detailed study of these representations, see [21]. They are also called the *spherical* (*non-unitary*) *principal series* of G.

The reason for putting ρ in the parameter of the representation is to introduce more symmetry in the formulation of the following *duality* result. For f and φ in $\mathcal{C}^\infty(S)$, and for any $g \in G$,

$$\int_S \pi_\lambda(g) f(x)\varphi(x)\, d\sigma(x) = \int_S f(y)\pi_{-\lambda}\big(g^{-1}\big)\varphi(x)\, d\sigma(x). \tag{5}$$

This is a consequence of the change of variable formula (3).

The operators K_α (formally) introduced above are intertwining operators for the spherical principal series (a special case of the *Knapp–Stein intertwining operators*). In fact, for any $g \in G$,

$$K_{-(n-1)+2\lambda} \circ \pi_\lambda(g) = \pi_{-\lambda}(g) \circ K_{-(n-1)+2\lambda}. \tag{6}$$

1.3. Conformally invariant trilinear forms: the generic case

Now recall the construction and results of [2]. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be in $\mathbb{C}^3$. Consider the kernel $K_{\boldsymbol{\alpha}}$ on $S \times S \times S$ defined by

$$K_{\boldsymbol{\alpha}}(x_1, x_2, x_3) = k_{\alpha_1}(x_2, x_3)k_{\alpha_2}(x_3, x_1)k_{\alpha_3}(x_1, x_2)$$

and the associated trilinear form $\mathcal{K}_{\boldsymbol{\alpha}}$ defined by

$$\mathcal{K}_{\boldsymbol{\alpha}}(f_1, f_2, f_3) = \int_{S\times S\times S} K_{\boldsymbol{\alpha}}(x_1, x_2, x_3) f_1(x_1) f_2(x_2) f_3(x_3)\, d\sigma(x_1)\, d\sigma(x_2)\, d\sigma(x_3).$$

Proposition 1.2. *Let* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ *and* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ *in* $\mathbb{C}^3$ *be related by*

$$\begin{aligned} \alpha_1 &= -\rho - \lambda_1 + \lambda_2 + \lambda_3, \\ \alpha_2 &= -\rho + \lambda_1 - \lambda_2 + \lambda_3, \\ \alpha_3 &= -\rho + \lambda_1 + \lambda_2 - \lambda_3. \end{aligned} \tag{7}$$

Let f_1, f_2, f_3 *be three functions in* $\mathcal{C}^\infty(S)$.

(i) *Assume*

$$\operatorname{Re}\alpha_j > -(n-1), \quad j = 1, 2, 3, \qquad \operatorname{Re}(\alpha_1 + \alpha_2 + \alpha_3) > -2(n-1).$$

Then the integral $\mathcal{K}_{\boldsymbol{\alpha}}(f_1, f_2, f_3)$ *is absolutely convergent.*

(ii) *The mapping* $\boldsymbol{\alpha} \mapsto \mathcal{K}_{\boldsymbol{\alpha}}(f_1, f_2, f_3)$ *can be extended as a meromorphic function, with simple poles along the planes (in* $\mathbb{C}^3$*)*

$$\begin{aligned} \alpha_1 &= -(n-1) - 2k_1, \quad k_1 \in \mathbb{N}, \\ \alpha_2 &= -(n-1) - 2k_2, \quad k_2 \in \mathbb{N}, \\ \alpha_3 &= -(n-1) - 2k_3, \quad k_3 \in \mathbb{N}, \\ \alpha_1 + \alpha_2 + \alpha_3 &= -2(n-1) - 2k, \quad k \in \mathbb{N}. \end{aligned} \tag{8}$$

(iii) *Assume that $\boldsymbol{\alpha}$ does not satisfy any of Eqs.* (8) (*i.e. is not a pole*). *The trilinear form $\mathcal{K}_{\boldsymbol{\alpha}}$ defined by* (ii) *is invariant w.r.t. the representations $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$. Moreover, $\mathcal{K}_{\boldsymbol{\alpha}}$ is* (*up to a constant*) *the unique trilinear form on $\mathcal{C}^\infty(S)$ which is invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$.*

The set of poles is a collection of planes in $\mathbb{C}^3$. In the present article, we consider only *generic poles*, i.e. which lie in a unique plane of poles. A pole $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ will be said *of type I* if there exist $j \in \{1, 2, 3\}$ and an integer k_j such that

$$\alpha_j = -(n-1) - 2k_j \tag{9}$$

and of *type II* if there exists an integer k such that

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k. \tag{10}$$

2. Residues at poles of type I

2.1. Construction of singular invariant trilinear forms

Let k be in $\mathbb{N}$. There exists a canonical differential operator Δ_k of order $2k$ on S covariant with respect to (π_{-k}, π_k), i.e. which satisfies, for any g in G

$$\Delta_k \circ \pi_{-k}(g) = \pi_k(g) \circ \Delta_k.$$

Its construction, using the meromorphic continuation of the distribution $|\mathbf{1} - x|^s$ is recalled in Appendix A.

Set for convenience $\alpha_3 = -(n-1) - 2k$ (notation will be explained later in this section).

For α_1, α_2 in $\mathbb{C}^2$, define $\mathcal{T}_k = \mathcal{T}_{(\alpha_1, \alpha_2, -(n-1)-2k)}$ on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ by

$$\mathcal{T}_k(f_1, f_2, f_3) = \int_{S\times S} f_3(x_3) f_2(x) \Delta_k\big[f_1(.)|x_3 - .|^{\alpha_2}\big](x)|x - x_3|^{\alpha_1}\, d\sigma(x)\, d\sigma(x_3).$$

Proposition 2.1. *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3 = -\rho - 2k)$ and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ satisfying the relations* (7). *Then*

$$\mathcal{T}_k\big(\pi_{\lambda_1}(g) f_1, \pi_{\lambda_2}(g) f_2, \pi_{\lambda_3}(g) f_3\big) = \mathcal{T}_k(f_1, f_2, f_3)$$

for any g in G and $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$, whenever the integrals make sense.

Proof. We first need a technical lemma. It will be convenient to set, for f_1 in $\mathcal{C}^\infty(S)$ and x_3 in S,

$$F_{x_3}[f_1](x) = f_1(x)|x_3 - x|^{\alpha_2}.$$

Lemma 2.1. *For any g in G, f_1 in $\mathcal{C}^\infty(S)$ and x_3 in S,*

$$F_{x_3}\big[\pi_{\lambda_1}(g) f_1\big] = \kappa(g, y_3)^{\frac{\alpha_2}{2}} \pi_{-k}(g) F_{y_3}[f_1], \tag{11}$$

where $x_3 = g(y_3)$.

Proof. Let x be in S.

$$\begin{aligned} LHS \text{ at } x &= |x_3 - x|^{\alpha_2} f_1\big(g^{-1}(x)\big)\kappa\big(g^{-1}, x\big)^{\rho+\lambda_1} \\ &= f_1\big(g^{-1}(x)\big)\kappa(g, y_3)^{\frac{\alpha_2}{2}}\big|y_3 - g^{-1}(x)\big|^{\alpha_2}\kappa\big(g^{-1}, x\big)^{-\frac{\alpha_2}{2}+\rho+\lambda_1} \end{aligned}$$

using (4). Now, using (7) and the specific value of α_3,

$$-\frac{\alpha_2}{2} + \rho + \lambda_1 = \rho - k,$$

so that

$$LHS \text{ at } x = f_1\big(g^{-1}(x)\big)\kappa(g, y_3)^{\frac{\alpha_2}{2}}\big|y_3 - g^{-1}(x)\big|^{\alpha_2}\kappa\big(g^{-1}, x\big)^{\rho-k}.$$

On the other hand,

$$\begin{aligned} RHS \text{ at } x &= \kappa(g, y_3)^{\frac{\alpha_2}{2}}\kappa\big(g^{-1}, x\big)^{\rho-k} F_{y_3}[f_1]\big(g^{-1}(x)\big) \\ &= \kappa(g, y_3)^{\frac{\alpha_2}{2}}\big|y_3 - g^{-1}(x)\big|^{\alpha_2} f_1\big(g^{-1}(x)\big)\kappa\big(g^{-1}, x\big)^{\rho-k}, \end{aligned}$$

from which (11) follows. □

With the notation introduced previously,

$$\mathcal{T}_k(f_1, f_2, f_3) = \iint f_3(x_3) f_2(x)\Delta_k F_{x_3}[f_1](x)|x - x_3|^{\alpha_1}\, d\sigma(x)\, d\sigma(x_3),$$

so that

$$\begin{aligned} &\mathcal{T}_k\big(\pi_{\lambda_1}(g) f_1, \pi_{\lambda_2}(g) f_2, \pi_{\lambda_3}(g) f_3\big) \\ &\quad = \iint f_3\big(g^{-1}(x_3)\big) f_2\big(g^{-1}(x)\big)\Delta_k\big\{F_{x_3}\big[\pi_{\lambda_1}(g) f_1\big]\big\}(x)|x - x_3|^{\alpha_1} \\ &\qquad \cdots \kappa\big(g^{-1}, x\big)^{\rho+\lambda_2}\kappa\big(g^{-1}, x_3\big)^{\rho+\lambda_3}\, d\sigma(x)\, d\sigma(x_3). \end{aligned}$$

Now use (11) and (39), make the change of variables $x = g(y)$ and $x_3 = g(y_3)$ to get

$$\begin{aligned} &\mathcal{T}_k\big(\pi_{\lambda_1}(g) f_1, \pi_{\lambda_2}(g) f_2, \pi_{\lambda_3}(g) f_3\big) \\ &\quad = \iint f_3(y_3) f_2(y)\pi_k(g)\Delta_k\big\{F_{y_3}[f_1]\big\}\big(g(y)\big)|y - y_3|^{\alpha_1} \\ &\qquad \cdots \kappa(g, y_3)^{\frac{\alpha_2}{2}+\frac{\alpha_1}{2}-\rho-\lambda_3+2\rho}\kappa(g, y)^{\frac{\alpha_1}{2}-\rho-\lambda_2+2\rho}\, d\sigma(y)\, d\sigma(y_3) \\ &\quad = \iint f_3(y_3) f_2(y)\Delta_k F_{y_3}[f_1](y)|y - y_3|^{\alpha_1} \\ &\qquad \cdots \kappa(g, y_3)^{\frac{\alpha_2}{2}+\frac{\alpha_1}{2}-\rho-\lambda_3+2\rho}\kappa(g, y)^{\frac{\alpha_1}{2}-\rho-\lambda_2+2\rho-\rho-k}\, d\sigma(y)\, d\sigma(y_3). \end{aligned}$$

Now

$$\frac{\alpha_2}{2} + \frac{\alpha_1}{2} - \rho - \lambda_3 + 2\rho = 0,$$
$$\frac{\alpha_1}{2} - \rho - \lambda_2 + 2\rho - \rho - k = 0,$$

where in the second line we use the condition $\alpha_3 = -(n-1) - 2k$. So the last integral reduces to

$$\iint f_3(y_3) f_2(y) \Delta_k F_{y_3}[f_1](y) |y - y_3|^{\alpha_1} \, d\sigma(y) \, d\sigma(y_3),$$

which shows Proposition 2.1. □

Having proved (formally) the invariance of the form $\mathcal{T}_k$, we now study the convergence of the integral and its meromorphic continuation.

Theorem 2.1. *Let k be in $\mathbb{N}$. Then the trilinear form $\mathcal{T}_k = \mathcal{T}_{\alpha_1,\alpha_2,-\rho-2k}$ originally defined as a convergent integral for $\operatorname{Re}\alpha_1$ and $\operatorname{Re}\alpha_2$ large enough can be extended meromorphically to $\mathbb{C}^2$, with poles contained in the family of lines*

$$\alpha_1 + \alpha_2 = 2k - 2l, \quad l \in \mathbb{N}.$$

The trilinear form $\mathcal{T}_k$ is invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is associated to $\boldsymbol{\alpha}$ by the relations (7).

Proof. We first need a technical lemma.

Lemma 2.2. *Let φ be in $\mathcal{C}^\infty(S \times S)$. There exists a (unique) function $\psi(x, y, s)$ which is $\mathcal{C}^\infty$ in x and y and holomorphic in s such that, for all $x \neq y$ in $S \times S$*

$$\Delta_x\big[|x - y|^s \varphi(x, y)\big] = |x - y|^{s-2} \psi(x, y, s).$$

Moreover, the function ψ depends continuously on φ.

Proof. Recall the formula

$$\Delta(fg) = \Delta f g + 2\, \overrightarrow{\operatorname{grad}} f . \overrightarrow{\operatorname{grad}} g + f \Delta g.$$

By (33) and the fact that Δ commutes with the action of K

$$\Delta_x\big(|x - y|^s\big) = \left(-\frac{s}{2}\left(\frac{s}{2} + n - 2\right)|x - y|^2 + s(s + n - 3)\right)|x - y|^{s-2}.$$

Moreover,

$$\overrightarrow{\operatorname{grad}}_x\big(|x - y|^s\big) = \frac{s}{2}|x - y|^{s-2}\, \overrightarrow{\operatorname{grad}}_x\big(|x - y|^2\big).$$

The statement of the lemma is a consequence of these three formulæ. □

Now, by Eq. (36) and repeated uses of Lemma 2.2, it is possible to write

$$\Delta_k\big[f_1(.)|x_3-.|^{\alpha_2}\big](x)=\sum_{\ell=0}^{k}|x_3-x|^{\alpha_2-2\ell}\psi_\ell(x_3,x,\alpha_2)$$

where the ψ_ℓ are smooth functions on $S\times S$, holomorphic in α_2, depending continuously on f_1. Hence $\mathcal{T}_k(f_1,f_2,f_3)$ can be written as a sum of integrals of the form

$$\int_{S\times S} f_2(x)f_3(x_3)\psi_j(x_3,x,s)|x-x_3|^{\alpha_2+\alpha_1-2j}\,d\sigma(x_3)\,d\sigma(x)$$

where $j=0,1,\dots,k$, and ψ_j is a smooth function on $S\times S$, holomorphic in s and depending continuously on f_1. Such integrals can be meromorphically continued by use of Proposition A.7 and the localization of the poles also follows. □

2.2. *Residues at a pole of type I*

We now proceed to the determination of the residue of the form $\mathcal{K}_{\boldsymbol\alpha}$ at a pole of the form $\boldsymbol\alpha=(\alpha_1,\alpha_2,\alpha_3=-(n-1)-2k)$. Let R_k be the differential operator $\frac{\pi^\rho}{4^k\Gamma(\rho+k)\Gamma(k+1)}\Delta_k f$.

Theorem 2.2. *Let $\boldsymbol\alpha^0=(\alpha_1,\alpha_2,\alpha_3^0)$ be a generic pole in the plane $\alpha_3^0=-(n-1)-2k$. Then*

$$\begin{aligned}&\operatorname{Res}\big(K_{\boldsymbol\alpha}(f_1,f_2,f_3),\alpha^0\big)\\&\quad=\int_{S\times S} f_3(x_3)f_2(x)R_k\big[f_1(.)|x_3-.|^{\alpha_2}\big](x)|x-x_3|^{\alpha_1}\,d\sigma(x)\,d\sigma(x_3).\end{aligned}\tag{12}$$

More precisely, the right-hand side integral which converges for $\operatorname{Re}\alpha_1$ *and* $\operatorname{Re}\alpha_2$ *large enough can be extended meromorphically to $\mathbb{C}^2$ with poles contained in the lines*

$$\alpha_1+\alpha_2=-(n-1)+2k-2l,\quad l\in\mathbb{N}.$$

The left-hand side, a priori defined for α_1, α_2 outside of the lines

$$\alpha_1=-(n-1)-2k_1,\qquad \alpha_2=-(n-1)-2k_2,\qquad \alpha_1+\alpha_2=-(n-1)+2k-2l_3,$$

$k_1,k_2,l_3\in\mathbb{N}$, coincides with the right-hand side.

Proof. As R_k and Δ_k differ by a non-vanishing constant, the right-hand side of (12) depends meromorphically on (α_1,α_2) (see Theorem 2.1) so that, by properties of analytic continuation, it is enough to verify the equality when $\operatorname{Re}\alpha_1$ and $\operatorname{Re}\alpha_2$ are large enough. In this spirit, we have the following technical lemma.

Lemma 2.3. *Let f_3 be in $\mathcal{C}^\infty(S)$. For x_1, x_2 in S, let*

$$F_3(x_1,x_2)=\int_S f_3(x_3)|x_2-x_3|^{\alpha_1}|x_3-x_1|^{\alpha_2}\,d\sigma(x_3).\tag{13}$$

Let l be in $\mathbb{N}$. Then for $\operatorname{Re}\alpha_1$ *and* $\operatorname{Re}\alpha_2$ *large enough, the function F_3 is in $\mathcal{C}^l(S\times S)$, and the map $f_3 \mapsto F_3$ is continuous from $\mathcal{C}^\infty(S)$ to $\mathcal{C}^l(S\times S)$.*

Proof. Choose, at it is possible, $\operatorname{Re}\alpha_1$ and $\operatorname{Re}\alpha_2$ large enough, so that the kernel

$$|x_2-x_3|^{\alpha_1}|x_3-x_1|^{\alpha_2}$$

is, as a function of three variables, everywhere l-times continuously differentiable. Then the statement follows by the rule of differentiation under an integral sign and standard estimates. □

Now, let f_1, f_2, f_3 be three functions in $\mathcal{C}^\infty(S)$. Let $\operatorname{Re}\alpha_1$ and $\operatorname{Re}\alpha_2$ be large enough, so that the function F_3 defined by (13) is in $\mathcal{C}^l(S\times S)$ for some (large) l. For α_3 near α_3^0, but $\alpha_3 \neq \alpha_3^0$, the kernel $|x_1-x_2|^{\alpha_3}$ corresponds to a distribution on $S\times S$, depending meromorphically on α_3 and hence (see Proposition A.7 and the remark inside the proof) the expression

$$\int_{S\times S} f_1(x_1)f_2(x_2)F_3(x_1,x_2)|x_1-x_2|^{\alpha_3}\,d\sigma(x_1)\,d\sigma(x_2)$$

depends meromorphically on α_3. At $\alpha_3=-(n-1)-2k$, the expression has a simple pole and its residue is given by

$$\int_S R_k\big[f_1(.)f_2(x)F_3(.,x)\big](x)\,d\sigma(x).$$

Now, we differentiate under the integral sign (assuming, as it is possible, that F_3 is $2k$-times continuously differentiable), to get

$$\begin{aligned}&\operatorname{Res}\big(K_{\boldsymbol\alpha}(f_1,f_2,f_3),\boldsymbol\alpha^0\big)\\ &\quad=\int_{S\times S} f_2(x)f_3(x_3)R_k\big[f_1(.)|x_3-.|^{\alpha_2}\big](x)|x-x_3|^{\alpha_1}\,d\sigma(x_3)\,d\sigma(x).\end{aligned}$$

The equality (12) is thus proved for $\operatorname{Re}\alpha_1$ and $\operatorname{Re}\alpha_2$ large enough. □

The left-hand side of (12) is *a priori* defined in the intersection of the domain of definition of $K_{\boldsymbol\alpha}$ and the plane $\alpha_3=-(n-1)-2k$, that is to say on $\mathbb{C}^2$ only outside of the lines

$$\alpha_1=-(n-1)-2l_1,\qquad \alpha_2=-(n-1)-2l_2,\qquad \alpha_1+\alpha_2=-(n-1)+2k-2l_3,$$

where l_1, l_2, l_3 are in $\mathbb{N}$. A consequence of Theorem 2.2 is that it can be extended to a larger domain. This can be illustrated on the evaluation of $K_{\boldsymbol\alpha}$ for f_1, f_2, f_3 equal to the constant function 1. Then (see [3]), up to an entire function of $\boldsymbol\alpha$, $K_{\boldsymbol\alpha}(1,1,1)$ is equal to

$$\frac{\Gamma(\frac{\alpha_1+\alpha_2+\alpha_3}{2}+2\rho)\Gamma(\frac{\alpha_1}{2}+\rho)\Gamma(\frac{\alpha_2}{2}+\rho)\Gamma(\frac{\alpha_3}{2}+\rho)}{\Gamma(\frac{\alpha_2+\alpha_3}{2}+2\rho)\Gamma(\frac{\alpha_3+\alpha_1}{2}+2\rho)\Gamma(\frac{\alpha_1+\alpha_2}{2}+2\rho)}.$$

The residue at $\alpha_3 = -(n-1) - 2k$ is equal to

$$(-1)^k \frac{2}{k!} \frac{\Gamma(\frac{\alpha_1+\alpha_2}{2} - k + \rho)\Gamma(\frac{\alpha_1}{2} + \rho)\Gamma(\frac{\alpha_2}{2} + \rho)}{\Gamma(\frac{\alpha_2}{2} - k + \rho)\Gamma(\frac{\alpha_1}{2} - k + \rho)\Gamma(\frac{\alpha_1+\alpha_2}{2} + 2\rho)},$$

which equals

$$(-1)^k \frac{2}{k!}\left(\frac{\alpha_1}{2} + \rho - 1\right)\cdots\left(\frac{\alpha_1}{2} + \rho - k\right)\left(\frac{\alpha_2}{2} + \rho - 1\right)\cdots\left(\frac{\alpha_2}{2} + \rho - k\right)\frac{\Gamma(\frac{\alpha_1+\alpha_2}{2} - k + \rho)}{\Gamma(\frac{\alpha_1+\alpha_2}{2} + 2\rho)}$$

and this expression has simple poles contained in the lines

$$\alpha_1 + \alpha_2 = -(n-1) + 2k - 2l, \quad l \in \mathbb{Z}.$$

3. Residues at poles of type II

The computation of the residues at poles of type II is more difficult, and it forces us to use the realization of the representations π_λ in the *noncompact picture*. The usual change (from K/M to $\overline{N}$) is realized in our case by the *stereographic projection*. More precisely, for $y = (0, y_2, \dots, y_n) \in T_{\mathbf{1}}S$ define

$$c(y) = \begin{pmatrix} \frac{1-|y|^2}{1+|y|^2} \\ \frac{2y_2}{1+|y|^2} \\ \vdots \\ \frac{2y_n}{1+|y|^2} \end{pmatrix}.$$

Set $E = \mathbb{R}^{n-1}$ and view c as a diffeomorphism from E onto $S \setminus \{-\mathbf{1}\}$. Its inverse is (essentially) the stereographic projection with source at $-\mathbf{1}$ onto the tangent space at $\mathbf{1}$. It turns out that this map is conformal. In fact, for any $y \in E$, and any tangent vector $\eta \in \mathbb{R}^{n-1}$

$$\left|Dc(y)\eta\right| = 2\left(1 + |y|^2\right)^{-1}|\eta|.$$

Extending previous notation, set $\kappa(c, y) = 2(1 + |y|^2)^{-1}$. There is a global version of this infinitesimal result, namely

$$\left|c(y) - c(z)\right| = \kappa(c, y)^{1/2}|y - z|\kappa(c, z)^{1/2}. \tag{14}$$

By transferring the action of G to E, let for $g \in G$

$$\tilde{g} = c^{-1} \circ g \circ c.$$

This is a rational (not everywhere defined) conformal transformation of E. Denote by $\kappa(\tilde{g}, y)$ its conformal factor at $y \in E$. The spherical principal series can be realized in this noncompact setting. Let $\lambda \in \mathbb{C}$. For $g \in G$ and F a (smooth) function on E, let $\tilde{\pi}_\lambda(g)F$ be the function (almost everywhere) defined by

$$\tilde{\pi}_\lambda(g)F(y) = \kappa\big(\tilde{g}^{-1}, y\big)^{\rho+\lambda} F\big(\tilde{g}^{-1}(y)\big).$$

For f a function in $\mathcal{C}^\infty(S)$, let $\iota_\lambda f$ be the function defined on E by the formula[3]

$$\iota_\lambda f(y) = f\big(c(y)\big)\kappa(c, y)^{\rho+\lambda}. \tag{15}$$

The map ι_λ intertwines the representations π_λ and $\tilde{\pi}_\lambda$, as can be easily verified.

To get a well-defined representation, one can use the *Schwartz space of fast decreasing functions* $\mathcal{S}(E)$ and replace the action of the group G (which does not act on $\mathcal{S}(E)$) by the infinitesimal action of the Lie algebra $\mathfrak{g}$ (or rather the universal enveloping Lie algebra $\mathfrak{U}(g)$), once verified that this infinitesimal action is given by differential operators with *polynomial coefficients*. As the group G is connected, there is no loss in this modification. We skip the details.

Invariant trilinear forms can now be defined using the infinitesimal action, and they will correspond to *tempered* distributions on $E \times E \times E$. To describe them, let $\beta \in \mathbb{C}$, and set, for $x, y \in E$

$$l_\beta(x, y) = |x - y|^\beta.$$

Similarly, for $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathbb{C}^3$, let

$$l_{\boldsymbol{\beta}}(x_1, x_2, x_3) = l_{\beta_1}(x_2, x_3) l_{\beta_2}(x_3, x_1) l_{\beta_3}(x_1, x_2),$$

and let $\mathcal{L}_{\boldsymbol{\beta}}$ be the trilinear form defined on $\mathcal{S}(E) \times \mathcal{S}(E) \times \mathcal{S}(E)$ by

$$\mathcal{L}_{\boldsymbol{\beta}}(f_1, f_2, f_3) = \int_{E\times E\times E} f_1(x_1) f_2(x_2) f_3(x_3) l_{\boldsymbol{\beta}}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3.$$

Now the fact that the map c is conformal and formula (14) can be used to verify that

$$\mathcal{K}_{\boldsymbol{\beta}}(f_1, f_2, f_3) = \mathcal{L}_{\boldsymbol{\beta}}(\tilde{f}_1, \tilde{f}_2, \tilde{f}_3), \tag{16}$$

whenever the integrals make sense.

The meromorphic continuation of $L_{\boldsymbol{\beta}}$ as a tempered distribution on $E \times E \times E$ can be made along the same lines as done for the sphere case, and the theorem of generic existence and uniqueness of invariant trilinear forms on $\mathcal{S}(E) \times \mathcal{S}(E) \times \mathcal{S}(E)$ is almost verbatim identical to Proposition 1.2, the link between $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ and the parameters $(\lambda_1, \lambda_2, \lambda_3)$ is again given by

$$\begin{aligned}
\beta_1 &= -\rho - \lambda_1 + \lambda_2 + \lambda_3, \\
\beta_2 &= -\rho + \lambda_1 - \lambda_2 + \lambda_3, \\
\beta_3 &= -\rho + \lambda_1 + \lambda_2 - \lambda_3.
\end{aligned} \tag{17}$$

[3] The formula is natural if one recalls that f is in fact a section of a line bundle over S, depending on λ.

As we are interested in this section by poles of type II, we let, for $k \in \mathbb{N}$

$$\mathcal{H}_k = \{\boldsymbol{\beta} \in \mathbb{C}^3;\ \beta_1 + \beta_2 + \beta_3 = -2(n-1) - 2k\}.$$

The computation of residues will be obtained as follows: first we compute the residues at poles in the "first" plane $\mathcal{H}_0$, by a (rather) elementary computation. Then we prove a *Bernstein–Sato* identity for the kernel $l_{\boldsymbol{\beta}}$, which allows by induction on k to compute the residues at poles belonging to $\mathcal{H}_k$.

However, we find interesting to show *a priori* the link between these residues and conformally covariant bidifferential operators, which we explain in the first subsection.

3.1. Singular trilinear forms supported by the diagonal and covariant bidifferential operators

The trilinear form $\mathcal{L}_{\boldsymbol{\beta}}$ is easily seen to be invariant by diagonal translations, i.e. by mappings t_v, $v \in E$, where

$$t_v(x_1, x_2, x_3) = (x_1 + v, x_2 + v, x_3 + v).$$

This is just the invariance of the trilinear form under the translations, which are indeed conformal transformations of E. This invariance property will be preserved for any residue of the meromorphic trilinear form.

Moreover, the residue at a pole of type II (viewed as a distribution on $E \times E \times E$) is supported on the diagonal $\{(x, x, x),\ x \in E\}$ of $E \times E \times E$. This result is implicit in (but easy to deduce from) [2]. Now the translations operate *transitively* on this diagonal. This is the key to the main result of this subsection.

Let F be a finite-dimensional real vector space, and let V be a linear subspace of F. Let F' be the dual space of F, and let

$$V^{\perp} = \{\xi \in F';\ \xi_{|V} = 0\}.$$

Let u be a distribution on V. The assignment

$$\mathcal{C}_c^{\infty}(F) \ni \varphi \mapsto (u, \varphi_{|V})$$

defines a distribution on F, the natural extension of u, hereafter denoted by $\tilde{u}$. Clearly $\operatorname{supp}(\tilde{u}) = \operatorname{supp}(u) \subset V$. We now characterize the *wavefront set* of $\tilde{u}$ (cf. [10]).

Proposition 3.1. *Let u be in $\mathcal{D}'(V)$, and let $\tilde{u}$ be the associated distribution on F. Then*

$$\mathrm{WF}(\tilde{u}) = \{(x, \xi) \in \operatorname{supp}(u) \times (F' \setminus 0);\ \xi \in V^{\perp} \text{ or } (x, \xi_{|V}) \in \mathrm{WF}(u)\}. \tag{18}$$

Proof. Choose a subspace W such that $F = V \oplus W$. For $\xi \in F'$, let $\xi = \xi' + \xi''$, where $\xi' \in W^{\perp}$ and $\xi'' \in V^{\perp}$. Let φ be in $\mathcal{C}_c^{\infty}(F)$. Then $\varphi\tilde{u}$ is a distribution with compact support and its Fourier transform is given by

$$\mathcal{F}(\varphi\tilde{u})(\xi) = (u, e^{-i(\xi', .)}\varphi_{|V}).$$

Let (x_0, ξ_0) be in the set described by the RHS of (18). If $\xi_0' = 0$, then choose φ such that $\langle u, \varphi_{|V}\rangle \neq 0$ (which is always possible since x_0 belongs to $\operatorname{supp}(u)$), so that $\mathcal{F}(\varphi\tilde{u})(\xi)$ cannot decrease rapidly in a conic neighborhood of ξ_0. If (x_0, ξ_0') belongs to $\mathrm{WF}(u)$, $\mathcal{F}(\varphi\tilde{u})(\xi)$ cannot decrease rapidly on a conic neighborhood of ξ_0' in $W^\perp \setminus 0$, *a fortiori* on a conic neighborhood of ξ_0 in $F' \setminus 0$. Conversely, assume (x_0, ξ_0') does not belong to $\mathrm{WF}(u)$ and $\xi_0' \neq 0$. Then, for φ with a sufficiently small support near x_0 and for ξ in a (small enough) conic neighborhood of ξ_0, $\mathcal{F}(\varphi\tilde{u})(\xi)$ can be dominated by $C_N(1+|\xi'|)^{-N}$ for any integer N. But in a (sufficiently small) conic neighborhood of ξ_0' one has $|\xi| \leqslant C|\xi'|$ for some constant $C > 0$, so that $\mathcal{F}(\varphi\tilde{u})(\xi)$ is dominated by $C_N(1+|\xi|)^{-N}$. Hence $(x_0, \xi_0) \notin \mathrm{WF}(\tilde{u})$. □

To further investigate distributions supported on V, one needs to introduce *normal derivatives*. Fix a splitting $F = V \oplus W$ as above, and choose coordinates $w_1, w_2, \dots, w_p$ on W, which can be regarded as (a partial set of) coordinates on F by extending them by 0 on V. Let $I = (i_1, \dots, i_p)$ be a p-tuple of natural integers, let $|I| = i_1 + i_2 + \dots + i_p$. Let D_I be the operator (the D_I's are often referred to as *normal derivatives*), defined by

$$D_I\varphi(v) = \frac{\partial^{|I|}\varphi}{\partial w_1^{i_1} \cdots \partial w_p^{i_p}}(v),$$

mapping smooth functions on F to smooth functions on V. To any distribution u on V, one can associate the distribution $D_I\tilde{u}$ on F defined by

$$(-1)^{|I|}(D_I\tilde{u}, \varphi) = (u, D_I\varphi).$$

Observe that $\mathrm{WF}(D_I\tilde{u}) = \mathrm{WF}(\tilde{u})$. The inclusion $\subset$ is obvious, whereas the opposite inclusion is obtained by testing against functions φ of the form

$$\varphi(v, w) = \chi(v) w^I \psi(w), \tag{19}$$

where $\chi \in \mathcal{C}_c^\infty(V)$, $w^I = w_1^{i_1} \cdots w_d^{i_d}$ and ψ is a function in $\mathcal{C}_c^\infty(W)$ which is identically equal to 1 in a neighborhood of 0.

Now let U be a distribution on F, with $\operatorname{supp}(U) \subset V$. The structure theorem of L. Schwartz asserts that there exist distributions u_I on V such that

$$U = \sum_I D_I \tilde{u}_I,$$

where the sum is locally finite. Moreover, the u_I's are unique.

If all the distributions u_I are given by smooth densities, then from (18), $\mathrm{WF}(U) \subset F \times (V^\perp \setminus 0)$. The converse is true.

Proposition 3.2. *Let U be a distribution on F, supported in V, and assume that*

$$\mathrm{WF}(U) \subset V \times \left(V^\perp \setminus 0\right).$$

Then there exist smooth functions u_I on V such that, for any $\varphi \in \mathcal{C}_c^\infty(F)$

$$(U, \varphi) = \int_V \sum_I (-1)^{|I|} u_I(v) D_I \varphi(v)\, dv.$$

Proof. By the previous result, $U = \sum_I D_I \tilde{u}_I$, where u_I is some distribution on V. The assumption on the wavefront set of U, when tested against functions of the form given by (19) implies that, for each d-tuple I, $\mathrm{WF}(\tilde{u}_I) \subset V \times (V^\perp \setminus 0)$, which shows that $\mathrm{WF}(u_I) = \emptyset$ by Proposition 3.1. As the projection onto the first coordinate of the wavefront set is precisely the singular support, each u_I coincides with a smooth function on F. □

A *transverse differential operator D* is a mapping from $\mathcal{C}_c^\infty(F)$ in $\mathcal{C}_c^\infty(V)$ which is given by

$$D\varphi(v) = \sum_I a_I(v) D_I \varphi(v),$$

where D_I are the normal derivatives introduced earlier, and the a_I's are smooth functions on V. The sum is always assumed to be locally finite. Notice that the a_I are well determined, again by testing the operator against functions of the form given by (19). The previous proposition can be reformulated as: any distribution U supported on a linear subspace V, such that $\mathrm{WF}(U) \subset V \times (V^\perp \setminus 0)$ can be realized as

$$(U, \varphi) = \int_V D\varphi(v)\, dv,$$

for some transverse differential operator D. Moreover (once a splitting of F as $V \oplus W$ has been chosen) D is uniquely determined.

Invariance properties of a singular distribution are reflected in the associated transverse differential operator. Here is a special case, fitted for our needs.

Proposition 3.3. *Let U be in $\mathcal{D}'(F)$, supported on V. Assume that U is invariant under translations by elements of V. Then*

$$(U, \varphi) = \int_V D\varphi(v)\, dv,$$

where D is a transverse differential operator with constant coefficients.

Proof. Let $v \in V$, $v \neq 0$, and let X_v be the vector field on F which is constant and equal to v at each point of F. Then, the invariance property of U implies $X_v U = 0$. Hence, by [10, Theorem 8.3.1]

$$\mathrm{WF}(U) \subset \big\{(x, \xi) \in V \times \big(F' \setminus 0\big),\ \xi(v) = 0\big\}.$$

As v was arbitrary,

$$\mathrm{WF}(U) \subset \big\{(x, \xi) \in V \times \big(V^\perp \setminus 0\big)\big\}.$$

By Proposition 3.2, U is given by a transverse differential operator D, i.e.

$$(U,\varphi)=\int_V D\varphi(v)\,dv$$

where $D\varphi(v)=\sum_I a_I(v)D_I\varphi(v)$. Now, for any $v_0\in V$, X_{v_0} commutes with any D_I, such that, by integration by parts,

$$0=(U,X_{v_0}\varphi)=-\int_V\sum_I X_{v_0}a_I(v)D_I\varphi(v)\,dv.$$

Now fix a p-tuple I, and check this equality on functions of the form (19). It yields $X_{v_0}a_I=0$ for any $v_0\in V$ and hence a_I is a constant. □

We may now apply this general result to the residue at some pole of type II. A *bidifferential operator* on E is a linear map D from $\mathcal{C}_c^\infty(E\times E)$ into $\mathcal{C}_c^\infty(E)$ given by an expression of the form

$$Df(x)=\sum_{I,J}a_{I,J}(x)\frac{\partial^{|I|+|J|}}{\partial y^I\partial z^J}f(x,x),$$

where the I, J are multi-indices, and for each (I,J), $a_{I,J}$ is a smooth function on E, the sum being locally finite. Notice that the coefficients $a_{I,J}$ are uniquely determined, as can be shown by testing against functions of the form $(y-x_0)^I(z-x_0)^J\psi(y,z)$, where ψ is in $\mathcal{C}_c^\infty(E\times E)$ and is identically 1 in a neighborhood of (x_0,x_0).

Now, a bidifferential operator can be interpreted as a special case of transverse differential operator. In fact, let $F=E\times E\times E$, and let V be the diagonal in $E\times E\times E$. As transverse space, choose $W=\{(0,y,z),\ y,z\in E\}$. For these data, a transverse differential operator $\mathcal{D}$ is of the form

$$\mathcal{D}F(x,x,x)=\sum_{I,J}a_{I,J}(x)\frac{\partial^{|I|+|J|}}{\partial y^I\partial z^J}F(x,x,x).$$

On functions of the form $F=f\otimes g$ where $f\in\mathcal{C}_c^\infty(E)$ and $g\in\mathcal{C}_c^\infty(E\times E)$ (to mean $F(x,y,z)=f(x)g(y,z)$) this takes the form

$$\mathcal{D}F(x,x,x)=f(x)Dg(x)$$

where D is the bidifferential operator on $E\times E$ given by

$$Dg(x)=\sum_{I,J}a_{I,J}(x)\frac{\partial^{|I|+|J|}}{\partial y^I\partial z^J}g(x,x).$$

Now 3.3 leads to the following result concerning the residues at poles of type II.

Proposition 3.4. *Let $\boldsymbol{\beta}^0 \in \mathcal{H}_k$ for some $k \in \mathbb{N}$, and not belonging to any other plane of poles. Then there exists a unique bidifferential operator with constant coefficients $D_{\boldsymbol{\beta}^0}$ such that, for f in $\mathcal{C}_c^\infty(E)$ and $g \in \mathcal{C}_c^\infty(E \times E)$,*

$$\operatorname{Res}\big(\mathcal{L}_{\boldsymbol{\beta}}, \beta^0\big)(f \otimes g) = \int_E f(x) D_{\boldsymbol{\beta}^0} g(x)\, dx.$$

In proving this result, we only used the invariance of the residue distribution by the translations. We can now use the full invariance by G.

Definition 3.1. Let D be a bidifferential operator from $\mathcal{C}_c^\infty(E \times E)$ into $\mathcal{C}_c^\infty(E)$. Let λ, μ, ν be three complex numbers. Then D is said to be *covariant with respect to* $(\tilde{\pi}_\lambda \otimes \tilde{\pi}_\mu, \tilde{\pi}_\nu)$ if for any functions $f \in \mathcal{C}_c^\infty(E \times E)$ and g in a (small enough) neighborhood of the neutral element in G,

$$D\big(\tilde{\pi}_\lambda(g) \otimes \tilde{\pi}_\mu(g) f\big) = \tilde{\pi}_\nu(g)(Df).$$

Needless to say, this definition could be stated using the infinitesimal action of $\mathfrak{g}$ instead of the local action of G (compare with [17]).

Proposition 3.5. *Let D be a bidifferential operator from $\mathcal{C}_c^\infty(E \times E)$ into $\mathcal{C}_c^\infty(E)$. Let $\mathcal{L}$ be the continuous trilinear form defined for $f \in \mathcal{C}_c^\infty(E)$ and $g \in \mathcal{C}_c^\infty(E \times E)$ by*

$$\mathcal{L}(f \otimes g) = \int_E f(x) Dg(x)\, dx.$$

Let λ, μ, ν be three complex numbers. Then the form $\mathcal{L}$ is invariant with respect to $(\pi_\lambda, \pi_\mu, \pi_\nu)$ if and only if D is covariant with respect to $(\tilde{\pi}_\mu \otimes \tilde{\pi}_\nu, \tilde{\pi}_{-\lambda})$.

This is an immediate consequence of the duality formula (5).

Corollary 3.1. *Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathcal{H}_k$ for some $k \in \mathbb{N}$, and not belonging to any other plane of poles. Let $D_{\boldsymbol{\beta}}$ be the associated bidifferential operator given by Proposition* 3.4. *Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be given by Eqs.* (17). *Then the bidifferential operator $D_{\boldsymbol{\beta}}$ is covariant with respect to $(\tilde{\pi}_{\lambda_2} \otimes \tilde{\pi}_{\lambda_3}, \tilde{\pi}_{\lambda_2+\lambda_3+\rho+2k})$.*

This is a consequence of the previous proposition and of Proposition 3.4.

3.2. Residues along the "first" plane of poles of type II

We now give an elementary approach to the computation of a residue of type II, in the classical spirit of Gelfand and Shilov (see [8]). In particular, this approach gives the explicit value of the residue at a pole in the "first" plane of poles $\mathcal{H}_0$.

As already observed, the distribution $\mathcal{L}_{\boldsymbol{\beta}}$ is invariant by any diagonal translation. To take advantage of this, define for $\varphi \in \mathcal{C}_c^\infty(E \times E \times E)$

$$\Phi(y, z) = \int_E \varphi(v, y + v, z + v)\, dv. \tag{20}$$

For φ in $\mathcal{C}_c^\infty(E \times E \times E)$, the integral converges and defines a function Φ which belongs to $\mathcal{C}_c^\infty(E \times E)$. Moreover, the correspondence $\varphi \mapsto \Phi$ is continuous. Notice for further reference that

$$\Phi(0,0) = \int_E \varphi(x,x,x)\,dx. \tag{21}$$

Lemma 3.1. *Assume that* $\operatorname{Re}\beta_j > -(n-1)$, $j = 1,2,3$, *and* $\operatorname{Re}(\beta_1+\beta_2+\beta_3) > -2(n-1)$. *Then, for any* $\varphi \in \mathcal{C}_c^\infty(E \times E \times E)$

$$\mathcal{L}_{\boldsymbol{\beta}}(\varphi) = \int_{E\times E} |y|^{\beta_3}|z|^{\beta_2}|y-z|^{\beta_1}\Phi(y,z)\,dy\,dz.$$

Proof. The conditions on the parameter $\boldsymbol{\beta}$ guarantee the convergence of the integrals. The equality of the integrals is obtained through the affine change of variables

$$v = x_1, \qquad y = x_2 - x_1, \qquad z = x_3 - x_1. \qquad \square$$

Let

$$\Sigma = \big\{(\sigma,\tau) \in E \times E,\ |\sigma|^2 + |\tau|^2 = 1\big\}$$

be the unit sphere in $E \times E$, and denote by $d\mu$ the Lebesgue measure on Σ. Recall the integration formula in polar coordinates

$$\int_{E\times E} \Phi(y,z)\,dy\,dz = \int_0^\infty\int_\Sigma \Phi(r\sigma, r\tau)\,d\mu(\sigma,\tau)\,r^{2(n-1)-1}\,dr.$$

Lemma 3.2. *For* ψ *in* $\mathcal{C}_c^\infty(\Sigma)$ *let*

$$\mathcal{I}_{\boldsymbol{\beta}}(\psi) = \int_\Sigma |\sigma|^{\beta_3}|\tau|^{\beta_2}|\sigma-\tau|^{\beta_1}\psi(\sigma,\tau)\,d\mu(\sigma,\tau). \tag{22}$$

(i) *Assume that* $\operatorname{Re}\beta_j > -(n-1)$ *for* $j = 1,2,3$. *Then the integral* (22) *is convergent and defines a distribution* $\mathcal{I}_{\boldsymbol{\beta}}$.
(ii) *The map* $\boldsymbol{\beta} \mapsto \mathcal{I}_{\boldsymbol{\beta}}$ *can be extended meromorphically to* $\mathbb{C}^3$, *with simple poles along the family of planes given by the following equations*:

$$\beta_j = -(n-1) - 2k_j, \quad j = 1,2,3,\ k_j \in \mathbb{N}.$$

Proof. The three subsets of Σ

$$\big\{(\sigma,\tau) \in \Sigma;\ \sigma = 0\big\}, \quad \big\{(\sigma,\tau) \in \Sigma;\ \tau = 0\big\}, \quad \big\{(\sigma,\tau) \in \Sigma;\ \sigma = \tau\big\}$$

are three *disjoint* submanifolds of dimension $(n-1)-1$ (hence of codimension $n-1$) in Σ. Recalling that we assumed $\operatorname{Re}\beta_j > -(n-1)$, for $j = 1, 2, 3$, the integrals

$$\int_\Sigma |\sigma|^{\beta_3}\, d\mu(\sigma,\tau), \qquad \int_\Sigma |\tau|^{\beta_2}\, d\mu(\sigma,\tau), \qquad \int_\Sigma |\sigma-\tau|^{\beta_1}\, d\mu(\sigma,\tau)$$

are convergent and hence the integral $\mathcal{I}_{\boldsymbol\beta}$ is convergent by applying a suitable argument involving a partition of unity. This shows (i). Similarly, the meromorphic extension and the location of poles (also the fact that the poles are simple) are classical and can be easily deduced from [8]. $\square$

Let Φ be a function in $\mathcal{C}_c^\infty(E\times E)$. For r in $\mathbb{R}$, let ψ_r be the function on Σ defined by

$$\psi_r(\sigma,\tau) = \Phi(r\sigma, r\tau), \quad (\sigma,\tau)\in\Sigma. \tag{23}$$

Then ψ_r belongs to $\mathcal{C}^\infty(\Sigma)$ and the map $(r,\Phi)\mapsto\psi_r$ is continuous from $\mathbb{R}\times\mathcal{C}_c^\infty(E\times E)$ to $\mathcal{C}^\infty(\Sigma)$.

Lemma 3.3. *Assume that* $\operatorname{Re}\beta_j > -(n-1)$, $j = 1,2,3$, *and* $\operatorname{Re}(\beta_1+\beta_2+\beta_3) > -2(n-1)$. *Then, for any* $\varphi\in\mathcal{C}_c^\infty(E\times E\times E)$

$$\mathcal{L}_{\boldsymbol\beta}(\varphi) = \int_0^\infty r^{2(n-1)-1+\beta_1+\beta_2+\beta_3}\mathcal{I}_{\boldsymbol\beta}\psi_r\, dr. \tag{24}$$

This is just using the formula for integration in polar coordinates.

Lemma 3.4. *Let γ be in $\mathcal{C}_c^\infty(\mathbb{R})$ and assume that γ is an* even *function. Then the integral*

$$I_s(\gamma) = \int_0^\infty r^s\gamma(r)\, dr = \frac{1}{2}\int_{-\infty}^{+\infty} |r|^s\gamma(r)\, dr$$

is convergent for $\operatorname{Re} s > -1$. *The map $s\mapsto I_s(\gamma)$ can be extended meromorphically to $\mathbb{C}$ with simple poles at $s = -1-2k$, $k\in\mathbb{N}$. Moreover, the residues at the poles are given by*

$$\operatorname{Res}\bigl(I_s(\gamma), -1-2k\bigr) = \frac{1}{\Gamma(2k+1)}\left(\frac{d}{dr}\right)^{2k}\gamma(0). \tag{25}$$

For a proof, see [8].

Theorem 3.1. *Let $k\in\mathbb{N}$ and let $\boldsymbol\beta^0 = (\beta_1^0, \beta_2^0, \beta_3^0)$ satisfy the following assumptions*:

(i) $\beta_1^0+\beta_2^0+\beta_3^0 = -2(n-1)-2k$;
(ii) $\beta_j^0 \notin -(n-1)-2\mathbb{N}$.

Let φ be a function in $\mathcal{C}_c^\infty(E \times E \times E)$ and form successively the functions Φ defined by (20) *and ψ_r defined by* (23). *The function $\beta \mapsto \mathcal{L}_{\boldsymbol\beta}(\varphi)$ has a residue at $\boldsymbol\beta^0$ given by*

$$\operatorname{Res}\big(\mathcal{L}_{\boldsymbol\beta}(\varphi), \boldsymbol\beta^0\big) = \frac{1}{\Gamma(2k+1)} \left(\frac{d}{dr}\right)^{2k}_{|r=0} \mathcal{I}_{\boldsymbol\beta^0}(\psi_r). \tag{26}$$

Proof. Observe that $\mathcal{I}_{\boldsymbol\beta}\psi_r$ is well defined (Lemma 3.2) and, as a function of r is easily seen to be in $\mathcal{C}_c^\infty(\mathbb{R})$. Moreover, the distribution $\mathcal{I}_{\boldsymbol\beta}$ is even, whereas $\psi_{-r}(\sigma,\tau) = \psi_r(-\sigma,-\tau)$, hence $\mathcal{I}_{\boldsymbol\beta}\psi_r$ is an even function of r. Now let $\gamma = \mathcal{I}_{\boldsymbol\beta}\psi_r$. Then (24) can be rewritten as $\mathcal{L}_{\boldsymbol\beta}(\varphi) = I_s(\gamma)$. Observe that $2(n-1) - 1 + \beta_1^0 + \beta_2^0 + \beta_3^0 = -1 - 2k$ and eventually apply (25) to conclude. □

The expression can be made explicit for a pole in the "first" plane of poles.

Proposition 3.6. *Assume that $\beta_1^0 + \beta_2^0 + \beta_3^0 = -2(n-1)$, and assume that $\beta_j \notin -(n-1) - 2\mathbb{N}$, $j = 1, 2$ or 3. Then, for any $f \in \mathcal{S}(E \times E \times E)$*

$$\operatorname{Res}\big(\mathcal{L}_{\boldsymbol\beta}(f), \boldsymbol\beta^0\big) = c_0\big(\boldsymbol\beta^0\big) \int_E f(x,x,x)\,dx \tag{27}$$

where

$$c_0\big(\boldsymbol\beta^0\big) = \left(\frac{\pi}{16\sqrt{2}}\right)^{n-1} \frac{\Gamma(n-1)}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(\frac{\beta_1^0}{2}+\rho)}{\Gamma(-\frac{\beta_1^0}{2}-\rho)} \frac{\Gamma(\frac{\beta_2^0}{2}+\rho)}{\Gamma(-\frac{\beta_2^0}{2}-\rho)} \frac{\Gamma(\frac{\beta_3^0}{2}+\rho)}{\Gamma(-\frac{\beta_3^0}{2}-\rho)}.$$

Proof. As $k = 0$, Eq. (26) shows that, up to a constant the residue equals $\Phi(0,0)$, which by (21) equals $\int_E f(x,x,x)\,dx$. Thus it is enough to verify (27) for *one* function. In the compact picture, there is a unique (up to a constant) K-invariant function namely the constant function equal to 1. In the noncompact picture, by the transformation property (15), it corresponds to the function

$$\tilde{1}(x) = 2^{\rho+\lambda}\big(1+|x|^2\big)^{-\rho-\lambda}.$$

Let

$$f_{\boldsymbol\lambda}(x_1,x_2,x_3) = 2^{3\rho+\lambda_1+\lambda_2+\lambda_3}\big(1+|x_1|^2\big)^{-\rho-\lambda_1}\big(1+|x_2|^2\big)^{-\rho-\lambda_2}\big(1+|x_3|^2\big)^{-\rho-\lambda_3}.$$

Now, let $\boldsymbol\beta$ be generic, and choose $\boldsymbol\lambda$ satisfying the relations (17). By use of (16), $\mathcal{L}_{\boldsymbol\beta}(f_{\boldsymbol\lambda}) = \mathcal{K}_{\boldsymbol\beta}(1)$, and the latter has been computed (for generic values of $\boldsymbol\beta$) in [6], and by a different method in [3] (see also [2]), so that

$$\mathcal{L}_{\boldsymbol\beta}(f_{\boldsymbol\lambda}) = \left(\frac{\sqrt{\pi}}{2}\right)^{3(n-1)} 2^{\beta_1+\beta_2+\beta_3+3\rho}$$
$$\cdots \frac{\Gamma(\frac{\beta_1+\beta_2+\beta_3}{2}+(n-1))\Gamma(\frac{\beta_1}{2}+\rho)\Gamma(\frac{\beta_2}{2}+\rho)\Gamma(\frac{\beta_3}{2}+\rho)}{\Gamma(\frac{\beta_2+\beta_3}{2}+\rho)\Gamma(\frac{\beta_3+\beta_1}{2}+\rho)\Gamma(\frac{\beta_1+\beta_2}{2}+\rho)}.$$

Hence, setting $f_0 = f_{\boldsymbol{\lambda}^0}$ where $\boldsymbol{\lambda}^0$ is associated to $\boldsymbol{\beta}^0$, we get

$$\operatorname{Res}\big(\mathcal{L}_{\boldsymbol{\beta}}, \boldsymbol{\beta}^0\big)(f_0) = \left(\frac{\sqrt{\pi}}{2}\right)^{3(n-1)} 2^{-\frac{n-1}{2}} \frac{\Gamma(\frac{\beta_1^0}{2}+\rho)\Gamma(\frac{\beta_2^0}{2}+\rho)\Gamma(\frac{\beta_3^0}{2}+\rho)}{\Gamma(\frac{\beta_2^0+\beta_3^0}{2}+\rho)\Gamma(\frac{\beta_3^0+\beta_1^0}{2}+\rho)\Gamma(\frac{\beta_1^0+\beta_2^0}{2}+\rho)}.$$

Now, for the right-hand side of (27), as

$$3\rho + \lambda_1^0 + \lambda_2^0 + \lambda_3^0 = 3(n-1) + \beta_1^0 + \beta_2^0 + \beta_3^0 = n-1,$$

$$\begin{aligned}\int_E f_0(x,x,x)\,dx &= 2^{n-1} \int_E \big(1+|x|^2\big)^{-(n-1)}\,dx \\ &= 2^{n-1} \operatorname{area}\big(S^{n-2}\big) \int_0^\infty \big(1+r^2\big)^{-(n-1)} r^{n-2}\,dr.\end{aligned}$$

Use the change of variables $r = \tan\theta$

$$\int_0^\infty \big(1+r^2\big)^{-(n-1)} r^{n-2}\,dr = \int_0^{\frac{\pi}{2}} \cos^{n-2}\theta \sin^{n-2}\theta\,d\theta = \frac{1}{2}\frac{\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)},$$

and recall that $\operatorname{area}(S^{n-2}) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ to get

$$\int_E f_0(x,x,x)\,dx = 2^{n-1}\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(n-1)}$$

from which the proposition follows. □

3.3. *Bernstein–Sato identity for the kernel $l_{\boldsymbol{\beta}}$*

A *Bernstein–Sato identity* (on the first parameter) is an identity of the form

$$B\ell_{\boldsymbol{\beta}+2_1} = b(\boldsymbol{\beta})\ell_{\boldsymbol{\beta}},$$

where $\boldsymbol{\beta} + 2_1 = (\beta_1 + 2, \beta_2, \beta_3)$, $B = B((x,y,z), \partial_x, \partial_y, \partial_z, \boldsymbol{\beta})$ is a differential operator with polynomial coefficients on $E \times E \times E$ and depending polynomially on $\boldsymbol{\beta}$, and b is a polynomial in three complex variables. Such identities are known to exist (see [19,20]), but are in general very difficult to find. It turns out that, in the case at hand, it is possible to explicitly construct such identities. The proof uses in a crucial way the covariance property of the kernel with respect to the conformal action of G on E.

We first present a formal approach, then explain the analytic details needed for the proof.

First introduce the *Knapp–Stein* intertwining operator I_ν formally defined by

$$I_\nu f(x) = \int_E |x-y|^{-(n-1)+\nu} f(y)\,dy.$$

The operator I_ν satisfies the important *intertwining property*, namely

$$I_{2\nu} \circ \tilde{\pi}_\nu(g) = \tilde{\pi}_{-\nu}(g) \circ I_{2\nu}.$$

Now consider the operator M given by

$$M\varphi(y,z) = |y-z|^2 \varphi(y,z). \tag{28}$$

Let λ, μ be two complex parameters. Then M intertwines the representations $(\tilde{\pi}_\lambda \otimes \tilde{\pi}_\mu)$ and $(\tilde{\pi}_{\lambda-1} \otimes \tilde{\pi}_{\mu-1})$ of G. In fact, let $g \in G$, and φ be a smooth function on $E \times E$. Suppose x, y are points in E where g is well defined. Then

$$\begin{aligned}
&\big[M \circ \tilde{\pi}_\lambda(g) \otimes \tilde{\pi}_\mu(g)\big]\varphi(y,z)\\
&\quad = |y-z|^2 \kappa\big(\tilde{g}^{-1}, y\big)^{\rho+\lambda} \kappa\big(\tilde{g}^{-1}, z\big)^{\rho+\mu} \varphi\big(\tilde{g}^{-1}(y), \tilde{g}^{-1}(z)\big)\\
&\quad = \big|\tilde{g}^{-1}(y) - \tilde{g}^{-1}(z)\big|^2 \kappa\big(\tilde{g}^{-1}, y\big)^{\rho+\lambda-1} \kappa\big(\tilde{g}^{-1}, z\big)^{\rho+\mu-1} \varphi\big(\tilde{g}^{-1}(y), \tilde{g}^{-1}(z)\big)\\
&\quad = \big[\tilde{\pi}_{\lambda-1}(g) \otimes \tilde{\pi}_{\mu-1}(g)\big](M\varphi)(y,z)
\end{aligned}$$

which is the intertwining property. For λ, μ two complex parameters, define the operator

$$N_{\lambda,\mu} = (I_{-2\lambda-2} \otimes I_{-2\mu-2}) \circ M \circ (I_{2\lambda} \otimes I_{2\mu}).$$

By construction $N_{\lambda,\mu}$ intertwines the representations $\tilde{\pi}_\lambda \otimes \tilde{\pi}_\mu$ and $\tilde{\pi}_{\lambda+1} \otimes \tilde{\pi}_{\mu+1}$. In a minute, we will show that $N_{\lambda,\mu}$ (and hence its transpose) is a *differential operator* on $E \times E$. Taking this for granted, let us continue with the formal deduction of the Bernstein–Sato identity.

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ be a generic triple in $\mathbb{C}^3$, let $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ be the triplet associated to $\boldsymbol{\lambda}$ through (17). Observe that $\boldsymbol{\beta} + 2_1$ is associated to the triple $(\lambda_1, \lambda_2 + 1, \lambda_3 + 1)$. Consider the continuous trilinear form $\mathcal{L}$ on $\mathcal{C}_c^\infty(E) \times \mathcal{C}_c^\infty(E) \times \mathcal{C}_c^\infty(E)$ given by

$$(f_1, f_2, f_3) \mapsto \mathcal{L}_{\boldsymbol{\beta}+2_1}\big(f_1 \otimes N_{\lambda_2,\lambda_3}(f_2 \otimes f_3)\big).$$

From the intertwining property of N_{λ_2,λ_3}

$$\begin{aligned}
&\mathcal{L}\big(\tilde{\pi}_{\lambda_1}(g) f_1, \tilde{\pi}_{\lambda_2}(g) f_2, \tilde{\pi}_{\lambda_3}(g) f_3\big)\\
&\quad = \mathcal{L}_{\boldsymbol{\beta}+2_1}\big(\tilde{\pi}_{\lambda_1}(g) f_1 \otimes N_{\lambda_2,\lambda_3}\big[\tilde{\pi}_{\lambda_2}(g) f_2 \otimes \tilde{\pi}_{\lambda_3}(g) f_3\big]\big)\\
&\quad = \mathcal{L}_{\boldsymbol{\beta}+2_1}\big(\tilde{\pi}_{\lambda_1}(g) f_1 \otimes \big[\tilde{\pi}_{\lambda_2+1}(g) \otimes \tilde{\pi}_{\lambda_3+1}(g)\big] \circ N_{\lambda_2,\lambda_3}[f_2 \otimes f_3]\big)\\
&\quad = \mathcal{L}_{\boldsymbol{\beta}+2_1}\big(f_1 \otimes N_{\lambda_2,\lambda_3}(f_2 \otimes f_3)\big)\\
&\quad = \mathcal{L}(f_1, f_2, f_3).
\end{aligned}$$

By the generic uniqueness result on the invariant trilinear form (see [2]), the form $\mathcal{L}$ has to be proportional to $\mathcal{L}_{\boldsymbol{\beta}}$. Hence there exists a constant $e = e(\boldsymbol{\beta})$ such that

$$(N_{\lambda_2,\lambda_3})^t(l_{\boldsymbol{\beta}+2_1}) = e(\boldsymbol{\beta})l_{\boldsymbol{\beta}} \tag{29}$$

which is the Bernstein–Sato identity we have been looking for.

To make this derivation rigorous, we need to consider spaces of functions on E (resp. on $E \times E$), on which the various operators are defined. Observe that a similar construction of operators could have been done on S (in the compact picture for the principal series). The spaces to be considered would be $\mathcal{C}^\infty(S)$ and $\mathcal{C}^\infty(S \times S)$. The corresponding intertwining operator (denoted by $K_{-(n-1)+2\lambda}$ in Appendix A) is well defined as a convolution on S with a distribution on S, hence maps $\mathcal{C}^\infty(S)$ into itself. As the completed inductive tensor product of $\mathcal{C}^\infty(S)$ can be identified with $\mathcal{C}^\infty(S \times S)$, the tensor product of the intertwining operators extends to a continuous map from $\mathcal{C}^\infty(S \times S)$ into itself. So that there is no difficulty in the compact picture, and the analog of the operator N_λ, μ (for λ, μ generic) is easy to define as an operator on $\mathcal{C}^\infty(S \times S)$. We now make the *ad hoc* definitions in order to transfer this result to the noncompact picture.

For $\lambda \in \mathbb{C}$, let $\mathcal{S}_\lambda$ be the space of functions $\tilde{f}$ on E which can be written as

$$\tilde{f}(x) = \iota_\lambda(f)(x) = f\big(c(x)\big)\kappa(c, x)^{\rho+\lambda},$$

where f is in $\mathcal{C}^\infty(S)$. Notice that the function f, if it exists, is unique. The space $\mathcal{S}_\lambda$ being isomorphic to $\mathcal{C}^\infty(S)$ can be equipped with the transferred topology of $\mathcal{C}^\infty(S)$.

For any g in G, we may define $\tilde{\pi}_\lambda(g)$ on $\mathcal{S}_\lambda$ by the formula

$$\tilde{\pi}_\lambda(g)\tilde{f} = \iota_\lambda\big(\pi_\lambda(g)f\big).$$

This defines a representation of G on $\mathcal{S}_\lambda$ and the map $f \mapsto \tilde{f}$ intertwines continuously the representations π_λ and $\tilde{\pi}_\lambda$. Notice the following inclusions

$$\mathcal{C}_c^\infty(E) \subset \mathcal{S}_\lambda \subset \mathcal{S}'(E).$$

Remark. The space $\mathcal{C}_c^\infty(E)$ is not dense in $\mathcal{S}_\lambda$. Fix some element k_0 in K such that $k_0(-\mathbf{1}) = \mathbf{1}$. Then a partition of unity argument on S shows that any function f in $\mathcal{S}_\lambda$ can be written as $f = f_1 + \tilde{\pi}_\lambda(k_0)f_2$ with $f_1, f_2 \in \mathcal{C}_c^\infty(E)$. So an operator which satisfies some covariance property under the action of G on $\mathcal{S}_\lambda$ is determined by its restriction to $\mathcal{C}_c^\infty(E)$.

In the compact picture, the intertwining operator (denoted by $K_{(n-1)+2\lambda}$ in (6)) can be seen as a convolution with a distribution on S (for generic values of λ, see Appendix A), hence it maps $\mathcal{C}^\infty(S)$ into itself. By transferring this result to the noncompact picture, the operator I_λ maps $\mathcal{S}_\lambda$ to $\mathcal{S}_{-\lambda}$. The tensor product $\mathcal{C}^\infty(S) \mathbin{\widehat{\otimes}} \mathcal{C}^\infty(S)$ can be realized as $\mathcal{C}^\infty(S \times S)$. For two complex numbers λ, μ, this gives a realization of the tensor product $\mathcal{S}_\lambda \mathbin{\widehat{\otimes}} \mathcal{S}_\mu$ as the space $\mathcal{S}_{\lambda,\mu}$ of functions $\tilde{f}$ on $E \times E$ which can be written as

$$\tilde{f}(x, y) = f\big(c(x), c(y)\big)\kappa(c, x)^{\rho+\lambda}\kappa(c, y)^{\rho+\mu}$$

for some function f in $\mathcal{C}^\infty(S \times S)$, the topology of $\mathcal{S}_{\lambda,\mu}$ being transferred from the topology of $\mathcal{C}^\infty(S \times S)$. Clearly,

$$\mathcal{C}_c^\infty(E \times E) \subset \mathcal{S}_{\lambda,\mu} \subset \mathcal{S}'(E \times E).$$

The following lemma is in the spirit of the previous remark.

Lemma 3.5. *Let $\lambda, \mu \in \mathbb{C}$, f be in $\mathcal{S}_{\lambda,\mu}$. Then there exist a finite family of elements $(k_j)_{j\in J}$ and a finite family of functions $(f_j)_{j\in J}$ in $\mathcal{C}_c^\infty(E\times E)$ such that*

$$f = \sum_{j\in J} \bigl[\tilde{\pi}_\lambda(k_j)\otimes \tilde{\pi}_\mu(k_j)\bigr] f_j.$$

Proof. For $r \geqslant 0$ and $x \in S$, let $B_r(x) = \{y\in S,\ |x-y|<r\}$. Fix $r>0$ and small enough. Choose a finite family of points $(x_i)_{i\in I}$ in S such that $S=\bigcup_{i\in I} B_r(x_i)$. Let i, j be in I. Then $\overline{B}_r(x_i)\cup \overline{B}_r(x_j)\neq S$ (as r is small), and so choose $x_{i,j}\in S$, $x_{ij}\notin \overline{B}_r(x_i)\cup\overline{B}_r(x_j)$. Now choose $k_{i,j}$ such that $k_{i,j}(-\mathbf{1}) = x_{ij}$. If $\varphi\in\mathcal{C}^\infty(S\times S)$ has its support contained in $B_r(x_i)\times B_r(x_j)$, then the function $f:(x,y)\mapsto \varphi(kx,ky)$ vanishes identically near $(-\mathbf{1},-\mathbf{1})$. Said differently, $\varphi = \pi_\lambda(k_{ij})\otimes\pi_\mu(k_{ij})f$, where f vanishes identically near $(-\mathbf{1},-\mathbf{1})$. Now let $J=I\times I$. A partition of unity argument (subordinated to the covering $S=\bigcup_{(i,j)\in J} B_r(x_i)\times B_r(x_j)$) shows that any function f in $\mathcal{C}^\infty(S\times S)$ can be written as

$$f = \sum_{(i,j)\in J} \bigl[\pi_\lambda(k_{ij})\otimes\pi_\mu(k_{ij})\bigr] f_{ij}$$

for some functions f_{ij} in $\mathcal{C}^\infty(S\times S)$ which vanish identically near $(-\mathbf{1},-\mathbf{1})$. Transfer to $E\times E$ to get the lemma. □

Let (λ,μ) be generic (i.e. such that the various intertwining operators are defined). The tensor product $I_{2\lambda}\otimes I_{2\mu}$ maps then continuously $\mathcal{S}_{\lambda,\mu}$ into $\mathcal{S}_{-\lambda,-\mu}$ and intertwines the representations $\tilde{\pi}_\lambda\otimes\tilde{\pi}_\mu$ with $\tilde{\pi}_{-\lambda}\otimes\tilde{\pi}_{-\mu}$. A similar statement holds for $I_{-2\lambda-2}\otimes I_{-2\mu-2}$. The operator M (multiplication by $|x-y|^2$) maps continuously $\mathcal{S}_{\lambda,\mu}$ into $\mathcal{S}_{\lambda-1,\mu-1}$, and so the operator $N_{\lambda,\mu}$ is a continuous operator from $\mathcal{S}_{\lambda,\mu}$ into $\mathcal{S}_{\lambda+1,\mu+1}$ and intertwines the representations $\tilde{\pi}_\lambda\otimes\tilde{\pi}_\mu$ and $\tilde{\pi}_{\lambda+1}\otimes\tilde{\pi}_{\mu+1}$. So we have set an analytical ground to justify the formal composition.

Proposition 3.7. *For $\lambda,\mu\in\mathbb{C}^2$, let $E_{\lambda,\mu}$ be the differential operator on $E\times E$ defined by*

$$\begin{aligned}E_{\lambda,\mu} &= |y-z|^2\Delta_y\Delta_z - 4\mu\sum_{j=1}^{n-1}(z_j-y_j)\frac{\partial}{\partial z_j}\Delta_y - 4\lambda\sum_{j=1}^{n-1}(y_j-z_j)\frac{\partial}{\partial y_j}\Delta_z\\ &\quad + 2\mu\bigl(2\mu+2-(n-1)\bigr)\Delta_y + 2\lambda\bigl(2\lambda+2-(n-1)\bigr)\Delta_z - 8\lambda\mu\sum_{j=1}^{n-1}\frac{\partial}{\partial y_j}\frac{\partial}{\partial z_j}.\end{aligned}$$

Its transpose $F_{\lambda,\mu} = E^t_{\lambda,\mu}$ is given by

$$\begin{aligned}F_{\lambda,\mu} &= |y-z|^2\Delta_y\Delta_z + 4(\mu+1)\sum_{j=1}^{n-1}(z_j-y_j)\frac{\partial}{\partial z_j}\Delta_y + 4(\lambda+1)\sum_{j=1}^{n-1}(y_j-z_j)\frac{\partial}{\partial y_j}\Delta_z\\ &\quad + 4(\mu+1)(\mu+\rho)\Delta_y + 4(\lambda+1)(\lambda+\rho)\Delta_z - 8(\lambda+1)(\mu+1)\sum_{j=1}^{n-1}\frac{\partial}{\partial y_j}\frac{\partial}{\partial z_j}.\end{aligned}$$

Assume that λ, μ are generic. Then the operator $F_{\lambda,\mu}$ maps $\mathcal{S}_{\lambda,\mu}$ into $\mathcal{S}_{\lambda+1,\mu+1}$ and

$$N_{\lambda,\mu} = c(\lambda,\mu) F_{\lambda,\mu},$$

where

$$c(\lambda,\mu) = \frac{\pi^{2(n-1)}}{16} \frac{\Gamma(\lambda)\Gamma(-\lambda-1)\Gamma(\mu)\Gamma(-\mu-1)}{\Gamma(\rho-\lambda)\Gamma(\rho+\lambda+1)\Gamma(\rho-\mu)\Gamma(\rho+\mu+1)}.$$

Proof. We will prove that $N(\lambda,\mu)$ and $c(\lambda,\mu)F_{\lambda,\mu}$ coincide on $\mathcal{C}_c^\infty(E\times E)$. The full statement is then a consequence of Lemma 3.5 and the intertwining property of $N_{\lambda,\mu}$.

Introduce the Fourier transform on E, defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_E e^{-i(\xi,x)} f(x)\,dx,$$

for any integrable function f, and extend it, as usual to $\mathcal{S}'(E)$. The Fourier transform on $E\times E$ is defined accordingly.

Observe that $\mathcal{S}_\lambda$ and $\mathcal{S}_{\lambda,\mu}$ are contained in the corresponding spaces of tempered distributions, on which the Fourier transform is defined. For sake of simplicity, we compute on tempered distributions as they were functions. Start with a function f in $\mathcal{C}_c^\infty(E\times E)$.

Observe that I_ν is a convolution operator with a tempered distribution, so that the Fourier transform of $I_\nu f$ is given by the product of the Fourier transform, i.e.

$$\mathcal{F}(I_\nu f)(\xi) = c(\nu)|\xi|^{-\nu}\hat{f}(\xi)$$

where

$$c(\nu) = 2^\nu \pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{(n-1)-\nu}{2})}$$

(see e.g. [8]).

Next, as M acts by multiplication by a polynomial, the Fourier transform of $M\varphi$ is given by

$$\mathcal{F}(M\varphi)(\xi,\eta) = (-\Delta_\xi + 2R - \Delta_\eta)\hat{\varphi}(\xi,\eta),$$

where Δ is the Laplacian on E and R is the differential operator on $E\times E$ defined by

$$R\varphi(\xi,\eta) = \sum_{j=1}^{n-1} \frac{\partial^2\varphi}{\partial\xi_j\partial\eta_j}.$$

To prove the formula, it is enough to prove it for functions $\varphi = f\otimes g$, where $f,g\in\mathcal{C}_c^\infty(E)$. Now

$$\frac{\partial}{\partial\xi_j}\big(|\xi|^{-2\lambda}\hat{f}(\xi)\big) = |\xi|^{-2\lambda}\frac{\partial\hat{f}}{\partial\xi_j} - 2\lambda|\xi|^{-2\lambda-2}\xi_j\hat{f}(\xi),$$

so that

$$\Delta_\xi\big(|\xi|^{-2\lambda}\hat{f}(\xi)\big) = |\xi|^{-2\lambda}\Delta\hat{f}(\xi) - 4\lambda|\xi|^{-2\lambda-2}\sum_{j=1}^{n-1}\xi_j\frac{\partial\hat{f}}{\partial\xi_j}$$
$$+ 2\lambda\big(2\lambda+2-(n-1)\big)|\xi|^{-2\lambda-2}\hat{f}(\xi)$$

and

$$R\big(|\xi|^{-2\lambda}\hat{f}(\xi)|\eta|^{-2\mu}\hat{g}(\eta)\big) = \sum_{j=1}^{n-1}|\xi|^{-2\lambda}\frac{\partial\hat{f}}{\partial\xi_j}|\eta|^{-2\mu}\frac{\partial\hat{g}}{\partial\eta_j} - 2\lambda\sum_{j=1}^{n-1}\xi_j|\xi|^{-2\lambda-2}\hat{f}(\xi)|\eta|^{-2\mu}\frac{\partial\hat{g}}{\partial\eta_j}$$
$$- 2\mu\sum_{j=1}^{n-1}|\xi|^{-2\lambda}\frac{\partial\hat{f}}{\partial\xi_j}\eta_j|\eta|^{-2\mu-2}\hat{g}(\eta)$$
$$+ 4\lambda\mu|\xi|^{-2\lambda-2}|\eta|^{-2\mu-2}\left(\sum_{j=1}^{n-1}\xi_j\eta_j\right)\hat{f}(\xi)\hat{g}(\eta).$$

Let apart the factor $c(2\lambda)c(2\mu)c(-2\lambda-2)c(-2\mu-2)$, the Fourier transform of $N_{\lambda,\mu}(f\otimes g)$ be given by

$$-|\xi|^2\Delta_\xi\hat{f}(\xi)|\eta|^2\hat{g}(\eta) + 4\lambda\sum_{j=1}^{n-1}\xi_j\frac{\partial\hat{f}}{\partial\xi_j}(\xi)|\eta|^2\hat{g}(\eta)$$
$$- 2\lambda\big(2\lambda+2-(n-1)\big)\hat{f}(\xi)|\eta|^2\hat{g}(\eta) + 2\sum_{j=1}^{n-1}|\xi|^2\frac{\partial\hat{f}}{\partial\xi_j}(\xi)|\eta|^2\frac{\partial\hat{g}}{\partial\eta_j}(\eta)$$
$$- 4\lambda\sum_{j=1}^{n-1}\xi_j\hat{f}(\xi)|\eta|^2\frac{\partial\hat{g}}{\partial\eta_j}(\eta) - 4\mu\sum_{j=1}^{n-1}|\xi|^2\frac{\partial\hat{f}}{\partial\xi_j}(\xi)\eta_j\hat{g}(\eta)$$
$$+ 8\lambda\mu\sum_{j=1}^{n-1}\xi_j\hat{f}(\xi)\eta_j\hat{g}(\eta) - |\xi|^2\hat{f}(\xi)|\eta|^2\Delta_\eta\hat{g}(\eta)$$
$$+ 4\mu\sum_{j=1}^{n-1}|\xi|^2\hat{f}(\xi)\eta_j\frac{\partial\hat{g}}{\partial\eta_j}(\eta) - 2\mu\big(2\mu+2-(n-1)\big)|\xi|^2\hat{f}(\xi)\hat{g}(\eta).$$

Now use the classical formulæ

$$\widehat{\frac{\partial f}{\partial y_j}}(\xi) = i\xi_j\hat{f}(\xi), \qquad \widehat{(y_j f)}(\xi) = i\frac{\partial\hat{f}}{\partial\xi_j}(\xi),$$
$$\widehat{\Delta f}(\xi) = -|\xi|^2\hat{f}(\xi), \qquad \widehat{|y|^2 f(y)}(\xi) = -\Delta\hat{f}(\xi)$$

to obtain the following expression for $N_{\lambda,\mu}(f\otimes g)$ (up to the factor $c(2\lambda)c(2\mu)c(-2\lambda-2)c(-2\mu-2)$):

$$\begin{aligned}
&\Delta_y\big(|y|^2 f\big)\Delta_z g + 4\lambda \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j}(y_j f)\Delta_z g \\
&\quad + 2\lambda\big(2\lambda + 2 - (n-1)\big) f \Delta_z g - 2\sum_{j=1}^{n-1} \Delta_y(y_j f)\Delta_z(z_j g) \\
&\quad - 4\lambda \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} f \Delta_z(z_j g) - 4\mu \sum_{j=1}^{n-1} \Delta_y(y_j f)\frac{\partial}{\partial z_j} g \\
&\quad - 8\lambda\mu \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} f \frac{\partial}{\partial z_j} g + \Delta_y f \Delta_z\big(|z|^2 g\big) \\
&\quad + 4\mu \sum_{j=1}^{n-1} \Delta_y f \frac{\partial}{\partial z_j}(z_j g) + 2\mu\big(2\mu + 2 - (n-1)\big)\Delta_y f g.
\end{aligned}$$

The final expression for $N_{\lambda,\mu}$ follows easily. □

As announced, E_{λ_2,λ_3} (being proportional to $N^t_{\lambda_2,\lambda_3}$) is a candidate for a Bernstein–Sato identity for the kernel $\ell_{\boldsymbol\beta}$ (the λ's being related to $\boldsymbol\beta$ by (17)). By brute force computation, the following identity is obtained.

Theorem 3.2 *(Bernstein–Sato identity). For $\boldsymbol\beta = (\beta_1, \beta_2, \beta_3)$ in $\mathbb{C}^3$, let $B_{\boldsymbol\beta}$ be the following differential operator on $E \times E$*

$$\begin{aligned}
B_{\boldsymbol\beta} = {} & |y-z|^2 \Delta_y \Delta_z + 2\big(\beta_3 + \beta_1 + (n-1)\big)\sum_{j=1}^{n-1}(z_j - y_j)\frac{\partial}{\partial y_j}\Delta_z \\
&+ 2\big(\beta_2 + \beta_1 + (n-1)\big)\sum_{j=1}^{n-1}(y_j - z_j)\frac{\partial}{\partial z_j}\Delta_y \\
&+ \big(\beta_3 + \beta_1 + (n-1)\big)(\beta_3 + \beta_1 + 2)\Delta_z \\
&+ \big(\beta_2 + \beta_1 + (n-1)\big)(\beta_2 + \beta_1 + 2)\Delta_y \\
&- 2\big(\beta_3 + \beta_1 + (n-1)\big)\big(\beta_2 + \beta_1 + (n-1)\big)\sum_{j=1}^{n-1}\frac{\partial^2}{\partial y_j \partial z_j}.
\end{aligned}$$

Then

$$B_{\boldsymbol\beta} l_{\boldsymbol\beta + 2_1} = b(\beta) l_{\boldsymbol\beta}$$

where

$$b(\boldsymbol\beta) = \big(\beta_1 + (n-1)\big)(\beta_1 + 2)\big(\beta_1 + \beta_2 + \beta_3 + 2(n-1)\big)\big(\beta_1 + \beta_2 + \beta_3 + (n-1) + 2\big).$$

3.4. Residues at poles of type II and covariant bidifferential operators

The Bernstein–Sato identity allows the computation of the residues of the distribution $\mathcal{L}_{\boldsymbol{\beta}}$ along the plane $\mathcal{H}_k$ by induction over k.

Proposition 3.8. *Let $\boldsymbol{\beta}^0$ be such that $\beta_1^0+\beta_2^0+\beta_3^0=-2(n-1)-2k-2$ for some $k\in\mathbb{N}$. Assume that $\beta_j\notin-(n-1)-2\mathbb{N}$ ($j=1,2,3$), and $\beta_1^0\neq-2$. Then, for $f\in\mathcal{C}_c^\infty(E)$ and $g\in\mathcal{C}_c^\infty(E\times E)$,*

$$\begin{aligned}
&\operatorname{Res}\big(\mathcal{L}_{\boldsymbol{\beta}},\boldsymbol{\beta}^0\big)(f\otimes g)\\
&\quad=\frac{1}{(2k+2)(2k+(n-1))(\beta_1^0+2)(\beta_1^0+(n-1))}\operatorname{Res}\big(\mathcal{L}_{\boldsymbol{\beta}},\boldsymbol{\beta}^0+2_1\big)\big(f\otimes B^t_{\boldsymbol{\beta}^0}g\big).
\end{aligned}$$

Proof. For generic values of $\boldsymbol{\beta}$, from the Bernstein–Sato identity,

$$\mathcal{L}_{\boldsymbol{\beta}}(f\otimes g)=(\ell_{\boldsymbol{\beta}},f\otimes g)=\frac{1}{b(\boldsymbol{\beta})}(B_{\boldsymbol{\beta}}\ell_{\boldsymbol{\beta}+2_1},f\otimes g)=\frac{1}{b(\boldsymbol{\beta})}\big(\ell_{\boldsymbol{\beta}+2_1},f\otimes B^t_{\boldsymbol{\beta}}g\big),$$

and compute the residue at $\boldsymbol{\beta}^0$ on both sides. $\square$

Let $C_{\boldsymbol{\beta}}=B^t_{\boldsymbol{\beta}}$. Except for the change of parameters, this is nothing but the operator $F_{\lambda\mu}$.

Proposition 3.9.

$$\begin{aligned}
C_{\boldsymbol{\beta}}=B^t_{\boldsymbol{\beta}}&=|y-z|^2\Delta_y\Delta_z+2\big(\beta_1+\beta_2+(n-1)+2\big)\sum_{j=1}^{n-1}(z_j-y_j)\frac{\partial}{\partial z_j}\Delta_y\\
&\quad+2\big(\beta_1+\beta_3+(n-1)+2\big)\sum_{j=1}^{n-1}(y_j-z_j)\frac{\partial}{\partial y_j}\Delta_z\\
&\quad+\big(\beta_1+\beta_2+2(n-1)\big)\big(\beta_1+\beta_2+(n-1)+2\big)\Delta_y\\
&\quad-2\big(\beta_1+\beta_2+(n-1)+2\big)\big(\beta_1+\beta_3+(n-1)+2\big)\sum_{j=1}^{n-1}\frac{\partial^2}{\partial y_j\partial z_j}\\
&\quad+\big(\beta_1+\beta_3+2(n-1)\big)\big(\beta_1+\beta_3+(n-1)+2\big)\Delta_z.
\end{aligned}$$

To write an expression for the residue at a pole in $\mathcal{H}_k$, where $k\in\mathbb{N}$, let us use the following convention: for $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)$ and $k\in\mathbb{N}$, let

$$\boldsymbol{\beta}-(2k)_1=(\beta_1-2k,\beta_2,\beta_3).$$

Now, for $\boldsymbol{\beta}\in\mathcal{H}_0$, define the differential operator $E^{(k)}_{\boldsymbol{\beta}}$ on $E\times E$ by

$$C^{(0)}_{\boldsymbol{\beta}}=\mathrm{Id},\qquad C^{(k)}_{\boldsymbol{\beta}}=C_{\boldsymbol{\beta}-2_1}\circ\cdots\circ C_{\boldsymbol{\beta}-(2k)_1}.$$

Also recall the *Pochhammer's symbol*, for a a complex number and $m \in \mathbb{N}$

$$(a)_m = a(a+1)(a+2)\cdots(a+m-1).$$

Theorem 3.3. *Let $\boldsymbol{\beta}^0 \in \mathcal{H}_0$, and let $k \in \mathbb{N}$. Assume that $\beta_j \notin -(n-1) - 2\mathbb{N}$ $(j = 1, 2, 3)$ and $\beta_1^0 \notin \{0, 2, \ldots, 2k-2\}$. Then*

$$\operatorname{Res}\big(\mathcal{L}_{\boldsymbol{\beta}}, \boldsymbol{\beta}^0 - (2k)_1\big)(f \otimes g) = c_k\big(\boldsymbol{\beta}^0\big) \int_E f(x)\big(C^{(k)}_{\boldsymbol{\beta}^0} g\big)(x, x)\, dx$$

where

$$c_k\big(\boldsymbol{\beta}^0\big) = \frac{1}{16^k}\frac{1}{k!}\frac{1}{(\rho)_k}\frac{1}{(-\frac{\beta_1^0}{2})_k}\frac{1}{(-\frac{\beta_1^0}{2} - \rho + 1)_k} c_0\big(\boldsymbol{\beta}^0\big).$$

The expression for the residue gives an explicit expression for the corresponding covariant bidifferential operators. Let λ, μ be in $\mathbb{C}$. For $k \in \mathbb{N}$, let $G^{(k)}_{\lambda\mu}$ be the bidifferential operator defined by

$$G^{(k)}_{\lambda\mu} f(x) = F_{\lambda+k-1\mu+k-1} \circ \cdots \circ F_{\lambda\mu} f(x, x). \tag{30}$$

Theorem 3.4. *Let λ, μ be in $\mathbb{C}$, and $k \in \mathbb{N}$. Then the operator $G^{(k)}_{\lambda\mu}$ is conformally covariant with respect to $(\tilde{\pi}_\lambda \otimes \tilde{\pi}_\mu, \tilde{\pi}_{\lambda+\mu+\rho+2k})$.*

Proof. Recall that the operator $F_{\lambda\mu}$ is a differential operator on $E \times E$, which is covariant with respect to $\tilde{\pi}_\lambda \otimes \tilde{\pi}_\mu, \tilde{\pi}_{\lambda+1} \otimes \tilde{\pi}_{\mu+1}$. So, by induction, $F_{\lambda+k-1\mu+k-1} \circ \cdots \circ F_{\lambda\mu}$ is covariant with respect to $(\tilde{\pi}_\lambda \otimes \tilde{\pi}_\mu, \tilde{\pi}_{\lambda+k} \otimes \tilde{\pi}_{\mu+k})$. Now the map

$$\mathcal{C}_c^\infty(E \times E) \ni f \mapsto \overline{f} \in \mathcal{C}_c^\infty(E),$$

where $\overline{f}(x) = f(x, x)$, is covariant with respect to $\tilde{\pi}_{\lambda+k} \otimes \tilde{\pi}_{\mu+k}, \pi_{\lambda+\mu+\rho+2k}$. The assertion follows. $\square$

For $k = 1$, one gets

$$G^{(1)}_{\lambda\mu} = 4(\mu+1)(\mu+\rho)\Delta_y - 8(\lambda+1)(\mu+1)R + 4(\lambda+1)(\lambda+\rho)\Delta_z. \tag{31}$$

Covariant bidifferential operators for the conformal group have been studied intensively, and the following result was obtained sometimes ago by V. Ovsienko and P. Redou (see [17]). They named these operators the *Rankin–Cohen brackets*, alluding to the case where the $n - 1 = 1$. Then the space $E \simeq \mathbb{R}$ can be seen as the boundary of the upper half-plane. The classical Rankin–Cohen operators are (explicit) holomorphic constant coefficients bidifferential operators, which are covariant under the action of $SL_2(\mathbb{R})$ on the upper half-plane. Taking their real counterpart on the boundary $\mathbb{R}$ yields covariant bidifferential operators. For a presentation of the Rankin–Cohen brackets in relation with the harmonic analysis of $SL_2(\mathbb{R})$, see [7].

Proposition 3.10. *Let k be a nonnegative integer, and let λ, μ be complex numbers.*

(i) *Assume that $\lambda, \mu \notin \{0, -1, -2, \dots, -(k-1)\}$. Then there exists a bidifferential operator $D^{(k)}_{\lambda,\mu}$ which is covariant with respect to $(\tilde{\pi}_\lambda \otimes \tilde{\pi}_\mu, \tilde{\pi}_{\lambda+\mu+\rho+2k})$.*
(ii) *Assume moreover that $\lambda, \mu \notin \{-\rho, -\rho-1, \dots, -\rho-(k-1)\}$. Then the operator is unique up to a constant.*

The operator $D^{(k)}_{\lambda,\mu}$ is explicitly described. The three fundamental bidifferential operators are Δ_y, Δ_z and the operator R defined for f in $\mathcal{C}^\infty(E \times E)$ by

$$R(f)(x) = \sum_{j=1}^{n-1} \frac{\partial^2 f}{\partial y_j \partial z_j}(x, x).$$

Then

$$D^{(k)}_{\lambda,\mu} = \sum_{r,s,t,r+s+t=k} c_{rst} \Delta_y^r R^s \Delta_z^t$$

where the c_{rst} are explicitly determined coefficients, depending on λ, μ and k, namely

$$\begin{aligned} c_{rst} = {} & \frac{(-1)^{t-r}}{2^r r!} \binom{r+s+t}{t} \frac{(s+1)_r}{(\lambda+1)_r} \\ & \times \sum_{p=0}^{r} \frac{r!t!}{p!} \frac{(\lambda+\rho+r-s+p)_{t-p}(\mu+\rho+s+2t)_{r-p}}{(\mu+1)_{t-p}} \end{aligned}$$

when $r \leqslant t$, and for $r \geqslant t$, $c_{rst}(\lambda, \mu) = c_{tsr}(\mu, \lambda)$.

For instance, if $k = 1$,

$$D^{(1)}_{\lambda,\mu} = -\frac{\mu+\rho}{\lambda+1} \Delta_y + 2R - \frac{\lambda+\rho}{\mu+1} \Delta_z. \tag{32}$$

This is to be compared with (31). For $k \geqslant 2$ and for generic (λ, μ), the uniqueness result guarantees that our bidifferential operator $G^{(k)}_{\lambda,\mu}$ is proportional to $D^{(k)}_{\lambda,\mu}$. However, it does not seem easy to compute the proportionality constant.

4. Final remarks

1. The construction of the covariant differential operator $N_{\lambda,\mu}$ admits a natural generalization. Let τ be the standard representation of G and τ' its dual representation. Choose highest weight vectors v and ϕ (with respect to some suitable ordering) for τ and τ', respectively. Then up to some constant, the multiplication by the matrix coefficient

$$g_1, g_2 \mapsto \langle \tau(g_1)v, \tau'(g_2)\phi \rangle$$

coincides with the operator M introduced in (28). Upon replacing τ by another irreducible finite-dimensional representation of G one obtains a multiplication operator that intertwines the tensor products of general (i.e. not necessarily spherical) principle series representations (see [1]). Using appropriate intertwining operators one obtains an operator analogous to $N_{\lambda,\mu}$. If this operator is always a differential operator is still an open question. We would also like to point out that the operators used by Oksak in his work [16] on invariant trilinear forms for $G = Sl_2(\mathbb{C}) \simeq Spin(3,1)$ can be constructed in this way. See also [12].

2. A byproduct of Proposition 3.6 is that the residue at a point $\boldsymbol{\beta} \in \mathcal{H}_0$ vanishes identically if $\frac{-\beta_1-(n-1)}{2} \in -\mathbb{N}$, i.e. if $\beta_1 \in -(n-1)+2m, m \in \mathbb{N}$. For such a value, observe that

$$\lambda_1 = \frac{\beta_2+\beta_3}{2} + \rho = \frac{-\beta_1 - 2(n-1)}{2} + \rho = -m.$$

Now for $m \geqslant 1$, recall the differential operator R_m (cf. Eq. (39)) on $\mathcal{C}_c^\infty(E)$ which is covariant w.r.t. (π_{-m}, π_m). Now let

$$\tilde{\boldsymbol{\beta}} = (\beta_1 - 2m, \beta_2 + 2m, \beta_3 + 2m),$$

so that

$$\tilde{\boldsymbol{\lambda}} = (m, \lambda_2, \lambda_3).$$

As $\tilde{\beta}_1 + \tilde{\beta}_2 + \tilde{\beta}_3 = -2(n-1) + 2m$, $\tilde{\beta}$ is no longer a pole. So the form $\mathcal{L}_{\tilde{\boldsymbol{\beta}}}$ is well defined, and the form

$$(f_1, f_2, f_3) \mapsto \mathcal{L}_{\tilde{\boldsymbol{\beta}}}(R_m f_1, f_2, f_3)$$

is invariant with respect to $(\pi_{-m}, \pi_{\lambda_2}, \pi_{\lambda_3})$. However, the relation $\lambda_1 + \lambda_2 + \lambda_3 = -\rho$ guarantees that the form

$$(f_1, f_2, f_3) \mapsto \int_E f_1(x) f_2(x) f_3(x)\, dx$$

is invariant under $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$. So, for $\lambda_1 = -m$ and (λ_2, λ_3) generic, we have produced *two* (linearly independent) trilinear invariant forms on $\mathcal{C}_c^\infty(E) \times \mathcal{C}_c^\infty(E) \times \mathcal{C}_c^\infty(E)$ w.r.t. $(\pi_{-m}, \pi_{\lambda_2}, \pi_{\lambda_3})$. Although we won't develop these aspects here, the same remark can be used to produce, for specific values of $\boldsymbol{\lambda}$, two (linearly independent) bidifferential operators covariant under the same actions of the conformal group. Notice that this is in concordance with the results and the philosophy of [13] and [14].

3. The relation between our formula for the "standard" covariant bidifferential operators (30) and the formulæ obtained in [17] or in [14] is still to be investigated, and the coefficients which relate them seem to be important. In the classical setting (i.e. for the original Rankin–Cohen brackets acting on the upper half-plane), much effort has been devoted to understand the structure of this family of operators (see [23,4,22,5,18]). We hope that our realization of these operators will add to the understanding of the family of generalized Rankin–Cohen brackets.

Appendix A. Meromorphic continuation of $|\mathbf{1}-x|^s$ and covariant differential operators

For s in $\mathbb{C}$, let $h_s(x)=|\mathbf{1}-x|^s$. This defines a smooth function on S outside of the point $\mathbf{1}$. For $\operatorname{Re} s>-(n-1)$, the function is integrable and will be considered as a distribution on S, still denoted by h_s. It depends holomorphically on s. We want to show that it can be extended meromorphically to $\mathbb{C}$. The main ingredient to do this is the *Bernstein–Sato identity* (*stricto sensu*, one should consider the Bernstein–Sato identity for the smooth function $|1-x|^2$ on S).

Proposition A.1. *The following identity holds on* $S\setminus\{\mathbf{1}\}$

$$\left[\Delta+\frac{s}{2}\left(\frac{s}{2}+n-2\right)\right]h_s=s(s+n-3)h_{s-2}. \tag{33}$$

Proof. A function f on S which is invariant under the subgroup M depends only on the distance from x to $\mathbf{1}$ and can be written in a unique way as $f(x)=\varphi(\theta)$, where $\theta=\arccos\langle x,\mathbf{1}\rangle$, and φ is a function defined on the interval $[0,\pi]$. As the Laplacian commutes to the rotations, the function Δf is also invariant under M, and the following relation holds:

$$\Delta f(x)=\varphi''(\theta)+\frac{(n-2)}{\tan\theta}\varphi'(\theta).$$

As $|\mathbf{1}-x|^2=2(1-\cos\theta)$, (33) follows easily. $\square$

Proposition A.2. *The function* $s\mapsto h_s$ *originally defined for* $\operatorname{Re} s>-(n-1)$ *can be extended as a* (*distribution-valued*) *meromorphic function on* $\mathbb{C}$, *with simple poles at* $s=-(n-1)-2k$, $k\in\mathbb{N}$.

The proof is standard and uses mainly integration by parts in the form

$$\left(h_s,\left(\Delta+\frac{s}{2}\left(\frac{s}{2}+n-2\right)\right)f\right)=\left(\left(\Delta+\frac{s}{2}\left(\frac{s}{2}+n-2\right)\right)h_s,f\right),$$

where f is any smooth test function. If the left-hand side is already defined, and $s(s+n-3)\neq 0$, it can be used to define $(h_{s-2}n,f)$. The meromorphic dependence on s is easy to verify.

A variant of Proposition A.2 will be used later on.

Proposition A.3. *Let* φ *be in* $\mathcal{C}^k(S)$ (*i.e.* k*-times continuously differentiable*), *then the function* $s\mapsto\int\varphi(x)h_s(x)\,d\sigma(x)$ *can be extended meromorphically in the open set* $\operatorname{Re}(s)<-\rho-2[\frac{k}{2}]$ *with at most simple poles at* $s=-\rho-2l$, $l\in\mathbb{N}$, $l<[\frac{k}{2}]$.

Denote by r_k the residue at $-(n-1)-2k$ of the distribution-valued function $s\mapsto h_s$.

Proposition A.4.

$$r_0:=\operatorname{Res}\bigl(h_s,-(n-1)\bigr)=\frac{\pi^\rho}{\Gamma(\rho)}\delta_{\mathbf{1}}. \tag{34}$$

Proof.

$$(h_s, f) = \int_S f(x)|\mathbf{1} - x|^s \, d\sigma(x)$$
$$= \int_S \big(f(x) - f(\mathbf{1})\big)|\mathbf{1} - x|^s \, d\sigma(x) + f(\mathbf{1}) \int_S |\mathbf{1} - x|^s \, d\sigma(x).$$

Now $|f(x) - f(\mathbf{1})| \leqslant C|x - \mathbf{1}|$, so that the first integral is absolutely convergent for s in a (small) neighborhood of $-(n-1)$. Hence, the contribution to the residue at $-(n-1)$ of the first integral is 0. Now

$$\int_S |\mathbf{1} - x|^s \, d\sigma(x) = 2^{n-1}\pi^\rho 2^s \frac{\Gamma(\frac{s}{2} + \rho)}{\Gamma(\frac{s}{2} + 2\rho)}.$$

The function on the right-hand side is meromorphic, has a simple pole at $s = -(n-1) = -2\rho$, with residue equal to $\frac{\pi^\rho}{\Gamma(\rho)}$. □

Proposition A.5.

$$r_1 := \operatorname{Res}\big(h_s, -(n-1) - 2\big) = \frac{\pi^\rho}{4\Gamma(\rho + 1)} \Delta_1 \delta_{\mathbf{1}}, \tag{35}$$

where

$$\Delta_1 = \Delta - \frac{1}{4}(n-1)(n-3)$$

is the conformal Laplacian *or* Yamabe operator *on S.*

Proof. The Bernstein–Sato identity (33) can be extended meromorphically to $\mathbb{C}$. Taking residue of both sides at $-(n-1)$ yields

$$2(n-1)\operatorname{Res}\big(h_s, -(n-1) - 2\big) = \left[\Delta - \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\right] \operatorname{Res}\big(h_s, -(n-1)\big),$$

which, via (34) gives (35). □

For any k in $\mathbb{N}$, introduce the differential operator Δ_k on S given by

$$\Delta_k = \prod_{j=1}^{k} \big(\Delta - (\rho + j - 1)(\rho - j)\big) = \prod_{j=1}^{k} \big(\Delta_1 + j(j-1)\big). \tag{36}$$

Observe that Δ_k is a polynomial of degree k in Δ which is of the form Δ^k + lower order terms. Observe that Δ_k is essentially self-adjoint.

Proposition A.6. *For k any positive integer,*

$$r_k = \frac{\pi^\rho}{4^k \Gamma(\rho+k)\Gamma(k+1)} \Delta_k \delta_{\mathbf{1}}.$$

Proof. Compute the residue at $s = -(n-1) - 2k$ of both sides of the Bernstein identity (33) to get

$$r_{k+1} = \frac{1}{4(\rho+k)(k+1)} \big(\Delta - (\rho+k)(\rho-k-1)\big) r_k.$$

Hence the result. □

Normalize the Haar measure dk on K such that, for any integrable function f on S

$$\int_K f(k\mathbf{1})\, dk = \int_S f(x)\, d\sigma(x).$$

Let f be a smooth function on S. Let $f^\sharp$ (resp. $g^\sharp$) be the function on K defined by $f^\sharp(k) = f(k\mathbf{1})$. For f and g two smooth functions on S, the convolution $f^\sharp \star g^\sharp$ is defined (as a function on K) by

$$\big(f^\sharp \star g^\sharp\big)(k) = \int_K f^\sharp\big(l^{-1}k\big) g^\sharp(l)\, dl.$$

It is a smooth function on K, which is right invariant by M, hence defines a function on S, denoted by $f \star g$. Suppose moreover that f is invariant by M, hence of the form $f(x) = F(\langle x, \mathbf{1}\rangle)$. Then, $f \star g$ is given by

$$(f \star g)(x) = \int_S F\big(\langle x, y\rangle\big) g(y)\, d\sigma(y). \tag{37}$$

The convolution of two functions in $\mathcal{C}^\infty(S)$ is in $\mathcal{C}^\infty(S)$ and the convolution can be extended to distributions on S.

For α a complex parameter, recall that k_α is defined on $S \times S$ by

$$k_\alpha(x, y) = |x - y|^\alpha,$$

and the corresponding operator (formally defined by)

$$K_\alpha f(x) = \int_S k_\alpha(x, y) f(y)\, d\sigma(y).$$

To make proper sense, the operator K_α can be reinterpreted as a convolution. As, for x, y in S, $|x - y| = 2(1 - \langle x, y\rangle)$, (37) implies

$$K_\alpha f = h_\alpha \star f.$$

Using the standard continuity properties of the convolution on S and the results of Section 2, it is easy to analytically continue the map $\alpha \mapsto K_\alpha$ on $\mathbb{C}$, with simple poles at $\alpha = -(n-1) - 2k$, $k \in \mathbb{N}$. The corresponding residues are easily computed. In fact, let $R_k = \operatorname{Res}(K_\alpha, -(n-1)-2k)$. Then

$$R_k f = r_k \star f = \frac{\pi^\rho}{4^k \Gamma(\rho+k)\Gamma(k+1)} \Delta_k f.$$

Now, for φ, ψ two functions in $\mathcal{C}^\infty(S)$,

$$(k_\alpha, \varphi \otimes \psi) = \int_{S\times S} k_\alpha(x,y)\varphi(x)\psi(y)\,d\sigma(x)\,d\sigma(y) = (K_\alpha\varphi, \psi).$$

As above, this bilinear form on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ can be meromorphically continued on $\mathbb{C}$, with same poles as above. But by Schwartz nuclear theorem, this is equivalent to say that the $\alpha \mapsto k_\alpha$ can be meromorphically continued as a distribution (on $S \times S$)-valued function, with same poles as above, and the residues are also easy to calculate.

Proposition A.7. *Let f be in $\mathcal{C}^\infty(S\times S)$. Then the expression*

$$\iint_{S\times S} f(x,y)|x-y|^\alpha\,d\sigma(x)\,d\sigma(y) \tag{38}$$

originally defined for $\operatorname{Re}\alpha$ *large enough can be continued meromorphically to $\mathbb{C}$, with simple poles at $\alpha = -(n-1) - 2k$, $k \in \mathbb{N}$. The residue at $\alpha = -(n-1) - 2k$ is given by*

$$\int_S R_k^{(1)} f(x,x)\,d\sigma(x),$$

where $R_k^{(1)}$ stands for the differential operator R_k acting on the first variable.

Remark A.1. We will need a slightly stronger version of this result. Let k be in $\mathbb{N}$. Then it is possible to choose ℓ large enough so that, if f is merely in $\mathcal{C}^\ell(S\times S)$, then the integral (38) can be meromorphically continued to $\operatorname{Re}(\alpha) > -(n-1) - 2k$. This result is proved exactly the same way, but using Proposition A.3 instead of Proposition A.2.

Remark A.2. The operator R_k is self-adjoint, so that the residue can also be written as $\int_S R_k^{(2)} f(x,x)\,d\sigma(x)$, that is by letting R_k act on the second variable.

An important property of the operator K_α is that it is an intertwining operator for the representations constructed earlier (it is the Knapp–Stein intertwining operator for the spherical principal series). In fact, by a change of variable (and meromorphic continuation), one shows that for any g in G

$$K_{-(n-1)+2\lambda} \circ \pi_\lambda(g) = \pi_{-\lambda}(g) \circ K_{-(n-1)+2\lambda}.$$

Hence, taking residue on both sides at $\lambda = -k$

$$R_k \circ \pi_{-k}(g) = \pi_k(g) \circ R_k \tag{39}$$

for any g in G. This shows that Δ_k is a covariant differential operator w.r.t. (π_{-k}, π_k).

For $k = 1$, (39) is a well-known property of the Yamabe operator on the sphere. For higher values of k, it corresponds to the conformal invariance of the Graham–Jenne–Mason–Sparling operators on the sphere (see [9,11]).

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